Online Appendix:

"Efficiency and Stability in Large Matching Markets"

(Not for Publication)

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January 22, 2018

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S.1 Preliminary Lemma

The following lemma will be used several times in what follows.

Lemma S1. *Fix any $\epsilon > 0$ and any $k = 1, ..., K$. There exists $\delta > 0$ small enough such that, with probability going to 1 as $n \to \infty$, for each agent in I, (i) each of his top $\delta|O_k|$ favorite objects in $O_{\geq k} := \cup_{\ell \geq k} O_\ell$ yields a payoff greater than $U(u_k, 1) - \epsilon$* and *(ii) every such object belongs to O_k; moreover, for each object in O, (iii) each of its top $\delta|I|$ individuals in I have priority scores of at least $V(1) - \epsilon$.*

For the proof, we first prove the following claim:

Claim: *Fix any $\tilde{\epsilon} > 0$. Let $\hat{I}$ and $\hat{O}$ be two sets such that both $|\hat{I}|$ and $|\hat{O}|$ are in between αn and n for some $\alpha > 0$. For each $i \in \hat{I}$, let X_i be the number of objects in $\hat{O}$ for which $\xi_{io} \geq 1 - \tilde{\epsilon}$. Then, for any $\delta < \tilde{\epsilon}$, $\Pr\{\exists i \text{ with } X_i \leq \delta\, |\hat{O}|\} \to 0$ as $n \to \infty$.*

Proof. X_i follows a binomial distribution $B(|\hat{O}|, \tilde{\epsilon})$ (recall that ξ_{io} follows a uniform distribution with support $[0,1]$). Hence:

$$\begin{aligned}\Pr\{\exists i \text{ with } X_i \le \delta|\hat{O}|\} \le & \quad \textstyle\sum_{i\in\hat{I}} \Pr\{X_i \le \delta|\hat{O}|\} \\ = & \quad |\hat{I}| \Pr\{X_i \le \delta|\hat{O}|\} \\ \le & \quad |\hat{I}| \tfrac{1}{2} \exp\left(-2\tfrac{(|\hat{O}|\tilde{\epsilon}-\delta|\hat{O}|)^2}{|\hat{O}|}\right) \\ = & \quad \tfrac{|\hat{I}|}{2\exp\left(2(\tilde{\epsilon}-\delta)^2|\hat{O}|\right)} \to 0,\end{aligned}$$

where the first inequality is by the union bound while the second inequality is by Hoeffding's inequality. ∎

Proof of Lemma S1. It should be clear that the third part of the statement can be proven using the same argument used for the second part. In addition, for a $\epsilon > 0$ sufficiently small that for each $k = 1, \dots, K-1$, $U(u_k, 1) - \epsilon > U(u_{k+1}, 1)$, objects in $O_{\ge k}$ that yield a payoff greater than $U(u_k, 1) - \epsilon$ can only be in O_k. Hence, the first part of the Lemma implies the second part. Thus, in the sequel, we only prove the first part of the statement.

Let us fix $\epsilon > 0$. By the continuity of $U(u_k, \cdot)$, there exists $\tilde{\epsilon} > 0$ such that $U(u_k, 1-\tilde{\epsilon}) > U(u_k, 1) - \epsilon$. By the above claim, with $\hat{I} := I$ and $\hat{O} := O_k$, there exists $\delta < \tilde{\epsilon}$ such that, with high probability, all individuals in I have at least $\delta|O_k|$ objects os in O_k for which $\xi_{io} > 1 - \tilde{\epsilon}$. By our choice of $\tilde{\epsilon}$, the payoffs that individuals enjoy for these objects must be higher than $U(u_k, 1) - \epsilon$. This implies that with probability going to 1, for every individual in I, his $\delta|O_k|$ most favorite objects in $O_{\ge k}$ yield a payoff greater than $U(u_k, 1) - \epsilon$, as claimed. ∎

S.2 Proof of Theorem 2

To begin, define a random set:

$$\hat{O} := \{o \in O_1 | o \text{ is assigned in TTC via long cycles}\}.$$

Lemma S2. *There exist $\gamma > 0, \delta > 0, N > 0$ such that*

$$\Pr\left\{\frac{|\hat{O}|}{n} > \delta\right\} > \gamma,$$

for all $n > N$.

Proof. Since the proof is rather long and requires several preliminary results, we organize the proof in Section S.3. ∎

For the next result, define

$$I_2 := \{i \in I | TTC(i) \in O_2\}$$

to be the (random) set of agents who are assigned objects in O_2 under TTC. We next establish that any randomly selected (unmatched) pair from $\hat{O}$ and I_2 forms an ϵ-block with positive probability for sufficiently small $\epsilon > 0$.

Lemma S3. *There exist $\varepsilon > 0, \zeta > 0$ such that, for any $\epsilon \in [0, \varepsilon)$:*

$$\Pr\left[\eta_{jo} \geq \eta_{TTC(o)o} + \epsilon \middle| o \in \hat{O}, j \in I_2\right] > \zeta.$$

PROOF. Note first that because the common value difference between O_1 and O_2 objects is large, if $o \in \hat{O} \subset O_1$ and $j \in I_2$, it must be the case that o does not point to j in the cycle to which o belongs under TTC (otherwise, if j is part of the cycle in which o is cleared, as $o \in O_1$, this means that j must be pointing to an object in O_1 when she is cleared, which contradicts $j \in I_2$). Note also that j is still in the market when o is cleared.

Define $E_1 := \{\eta_{jo} \geq \eta_{TTC(o)o}\} \wedge \{o \in \hat{O}\} \wedge \{j \in I_2\}$ and $E_2 := \{\eta_{jo} \leq \eta_{TTC(o)o}\} \wedge \{o \in \hat{O}\} \wedge \{j \in I_2\}$. We first show that $\Pr E_1 = \Pr E_2$.

Assume that given realizations $\boldsymbol{\xi} := (\xi_{io})_{io}$ and $\boldsymbol{\eta} := (\eta_{io})_{io}$, event E_1 occurs. Define $\hat{\boldsymbol{\eta}} := (\hat{\eta}_{io})_{io}$, where $\hat{\eta}_{jo} := \eta_{TTC(o)o}$ and $\hat{\eta}_{TTC(o)o} := \eta_{jo}$. $\hat{\boldsymbol{\eta}}$ and $\boldsymbol{\eta}$ coincide otherwise. It is easily verified that under the realizations $\boldsymbol{\xi}$ and $\hat{\boldsymbol{\eta}}$, event E_2 occurs. Indeed, that $\{\hat{\eta}_{jo} \leq \hat{\eta}_{TTC(o)o}\}$ holds true is trivial. Now, because, as we already claimed, under the realizations $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, j and $TTC(o)$ are never pointed to by o, when j and $TTC(o)$ are switched in o's priorities, by definition of TTC, o still belongs to the same cycle, and hence, TTC runs exactly in the same way. This shows that $\{o \in \hat{O}\} \wedge \{j \in I_2\}$ also holds true under the realizations $\boldsymbol{\xi}$ and $\hat{\boldsymbol{\eta}}$,

Since $\Pr(\boldsymbol{\xi}, \boldsymbol{\eta}) = \Pr(\boldsymbol{\xi}, \hat{\boldsymbol{\eta}})$, we can easily conclude that $\Pr E_1 = \Pr E_2$.

Next, let $E_\epsilon := \{\eta_{jo} \geq \eta_{TTC(o)o} + \epsilon\}$. Note:

$$\cup_{\epsilon > 0} E_\epsilon = \{\eta_{jo} > \eta_{TTC(o)o}\} =: E.$$

Because the distribution of η_{jo} has no atom, $\Pr\left[\cdot \middle| o \in \hat{O}, j \in I_2\right]$, the distribution of η_{jo} conditional on $o \in \hat{O}$ and $j \in I_2$ also has no atom ($\Pr(\eta_{jo} = \eta) = 0 \Rightarrow \Pr(\eta_{jo} = \eta \middle| o \in \hat{O}, j \in I_2) = 0$). Thus, we must have:

$$\Pr\left[E \middle| o \in \hat{O}, j \in I_2\right] = \Pr\left[\{\eta_{jo} \geq \eta_{TTC(o)o}\} \middle| o \in \hat{O}, j \in I_2\right] = \frac{1}{2}$$

where the last equality holds because $\Pr E_1 = \Pr E_2$.

As E_ϵ increases when ϵ decreases, combining the above, we obtain:[1]

$$\lim_{\epsilon\to 0}\Pr\left[E_\epsilon \,\middle|\, o\in\hat{O}, j\in I_2\right] = \Pr\left[\cup_{\epsilon>0}E_\epsilon \,\middle|\, o\in\hat{O}, j\in I_2\right] = \Pr\left[E \,\middle|\, o\in\hat{O}, j\in I_2\right] = \frac{1}{2}.$$

Thus, one can fix $\zeta\in(0,1/2)$ (which can be set arbitrarily close to $1/2$) and find $\varepsilon>0$ such that for any $\epsilon\in(0,\varepsilon)$, $\Pr\left[E_\epsilon \,\middle|\, o\in\hat{O}, j\in I_2\right]>\zeta$. ∎

Corollary S1. *For any $\epsilon>0$ sufficiently small, there exist $\zeta>0, N>0$ such that, for all $n>N$:*

$$\mathbb{E}\left[\frac{|\hat{I}_2^\epsilon(o)|}{n} \,\middle|\, o\in\hat{O}\right]\geq x_2\zeta$$

where $\hat{I}_2^\epsilon(o):=\{i\in I_2|\eta_{io}>\eta_{TTC(o)o}+\epsilon\}$.

PROOF. For any ϵ sufficiently small, we have $\zeta>0$ and $N>0$ such that for all $n>N$:

$$\begin{aligned}
\mathbb{E}\left[|\hat{I}_2^\epsilon(o)| \,\middle|\, o\in\hat{O}\right] &= \mathbb{E}\left[\sum_{i\in I_2}\mathbf{1}_{\{\eta_{io}>\eta_{TTC(o)o}+\epsilon\}} \,\middle|\, o\in\hat{O}\right]\\
&= \mathbb{E}_{I_2}\left(\mathbb{E}\left[\sum_{i\in I_2}\mathbf{1}_{\{\eta_{io}>\eta_{TTC(o)o}+\epsilon\}} \,\middle|\, o\in\hat{O}, I_2\right]\right)\\
&= \mathbb{E}_{I_2}\left(\sum_{i\in I_2}\mathbb{E}\left[\mathbf{1}_{\{\eta_{io}>\eta_{TTC(o)o}+\epsilon\}} \,\middle|\, o\in\hat{O}, I_2, i\in I_2\right]\right)\\
&= x_2 n\mathbb{E}_{I_2}\left(\mathbb{E}\left[\mathbf{1}_{\{\eta_{io}>\eta_{TTC(o)o}+\epsilon\}} \,\middle|\, o\in\hat{O}, I_2, i\in I_2\right]\right)\\
&= x_2 n\Pr(\eta_{io}>\eta_{TTC(o)o}+\epsilon \,|\, o\in\hat{O}, i\in I_2)\\
&= x_2 n\Pr(\eta_{io}\geq\eta_{TTC(o)o}+\epsilon \,|\, o\in\hat{O}, i\in I_2)\\
&\geq x_2\zeta n,
\end{aligned}$$

where the inequality follows from Lemma S3 . ∎

We are now ready to prove Theorem 2. The proof follows from Lemma S2 and Corollary S1. The former implies that as the economy grows, the expected number of objects in tier 1 assigned via long cycles remains significant. The latter implies that each of such objects finds many agents assigned by TTC to O_2 desirable for forming ϵ-blocks with. Specifically,

[1] Recall the following property. Let $\{E_n\}_n$ be an increasing sequence of events. Let $E:=\cup_n E_n$ be the limit of $\{E_n\}_n$. Then $\Pr(E)=\lim_{n\to\infty}\Pr(E_n)$.

for any sufficiently small $\epsilon \in (0, U(u_1, 0) - U(u_2, 1))$, we obtain that, for any $n > N$:

$$\begin{aligned}
\mathbb{E}\left[\frac{|J_\epsilon(TTC)|}{n(n-1)}\right] &\geq \mathbb{E}\left[\sum_{o\in\hat{O}} \frac{|\hat{I}_2^\epsilon(o)|}{n(n-1)}\right] \\
&\geq \Pr\{|\hat{O}| \geq \delta n\}\mathbb{E}\left[\sum_{o\in\hat{O}} \frac{|\hat{I}_2^\epsilon(o)|}{n(n-1)}\,\Big|\,|\hat{O}| \geq \delta n\right] \geq \gamma \mathbb{E}_{\hat{O}}\left(\mathbb{E}\left[\sum_{o\in\hat{O}} \frac{|\hat{I}_2^\epsilon(o)|}{n(n-1)}\,\Big|\,|\hat{O}| \geq \delta n, \hat{O}\right]\right) \\
&= \gamma \mathbb{E}_{\hat{O}}\left(\sum_{o\in\hat{O}} \mathbb{E}\left[\frac{|\hat{I}_2^\epsilon(o)|}{n(n-1)}\,\Big|\,|\hat{O}| \geq \delta n, \hat{O}, o \in \hat{O}\right]\right) \geq \gamma\delta n \mathbb{E}\left[\frac{|\hat{I}_2^\epsilon(o)|}{n(n-1)}\,\Big|\, o \in \hat{O}\right] \\
&\geq \gamma\delta \mathbb{E}\left[\frac{|\hat{I}_2^\epsilon(o)|}{n}\,\Big|\, o \in \hat{O}\right] \geq \gamma\delta\zeta x_2 > 0
\end{aligned}$$

where the first inequality follows from the observation that if $i \in \hat{I}_2^\epsilon(o)$ and $o \in \hat{O}$, then (i, o) ϵ-blocks TTC, while the penultimate inequality follows from Corollary S1.

S.3 Proof of Lemma S2

The proof of Lemma S2 requires a deeper understanding of the random structure of TTC. In the next section, we simply present the part of the results that are of direct use for the proof of Lemma S2. The result on the random structure of TTC is stated without proof, its proof can be found in Che and Tercieux (2017).

We need to begin with some preliminary definitions and results from random graph/mapping theory.

S.3.1 Preliminaries

To begin, recall that it is sufficient to consider the TTC assignment arising from the market consisting of the agents I and the objects O_1 in the top tier (recall that, irrespective of the realizations of the idiosyncratic values, all agents prefer every object in O_1 to any object in O_2). Hence, we shall simply consider an **unbalanced market** consisting of a set I of agents and a set O of objects such that (1) *the preferences of each side with respect to the other side are drawn iid uniformly,* and (2) $n = |I| \geq |O|$.

Consider any two finite sets I and O, with cardinalities $|I| = n, |O| = o$. A **bipartite digragh** $G = (I \times O, E)$ consists of vertices I and O on two separate sides and directed edges $E \subset (I \times O) \cup (O \times I)$, comprising ordered pairs of the form (i, o) or (o, i) (corresponding to edge originating from i and pointing to o and an edge from o to i, respectively). A **rooted**

tree is a bipartite digraph where all vertices have out-degree 1 except the root which has out-degree 0.[2] A **rooted forest** is a bipartite graph which consists of a collection of disjoint rooted trees. A **spanning rooted forest over** $I \cup O$ is a forest comprising vertices $I \cup O$. From now on, a spanning forest will be understood as being over $I \cup O$.

Consider an arbitrary mapping, $g : I \to O$ and $h : O \to I$, defined over our finite sets I and O. Note that such a mapping naturally induces a bipartite digraph with vertices $I \cup O$ and directed edges with the number of outgoing edges equal to the number of vertices, one for each vertex. In this digraph, $i \in I$ points to $g(i) \in O$ while $o \in O$ points to $h(o) \in I$. Such a mapping will be called a bipartite mapping. A **cycle** of a bipartite mapping is a cycle in the induced bipartite digraph, namely, distinct vertices $(i_1, o_1, ..., i_{k-1}, o_{k-1}, i_k)$ such that $g(i_j) = o_j, h(o_j) = i_{j+1}, j = 1, ..., k-1, i_k = i_1$. A **random bipartite mapping** selects a composite map $h \circ g$ uniformly from a set $\mathcal{H} \times \mathcal{G} = I^O \times O^I$ of all bipartite mappings. Note that a random bipartite mapping induces a random bipartite digraph consisting of vertices $I \cup O$ and directed edges emanating from vertices, one for each vertex. We say that a vertex in a digraph is **cyclic** if it is in a cycle of the digraph.

The following lemma states the number of cyclic vertices in a random bipartite digraph induced by a random bipartite mapping.

Lemma S4. *(Jaworski (1985), Corollary 3) The number q of the cyclic vertices in a random bipartite digraph induced by a random bipartite mapping $g : I \to O$ and $h : O \to I$ has an expected value of*

$$\mathbb{E}[q] := 2 \sum_{i=1}^{o} \frac{(o)_i (n)_i}{o^i n^i},$$

where $(x)_j := x(x-1) \cdots (x-j-1)$.

The next theorem on the random structure of TTC proves crucial for the proof of Lemma S2. Its proof is contained in Che and Tercieux (2017).

Theorem S1. *Suppose any round of TTC begins with n agents and o objects remaining in the market. Then, the probability that there are $m \le \min\{o, n\}$ agents assigned at the end of that round is*

$$p_{n,o;m} = \left(\frac{m}{(on)^{m+1}} \right) \left(\frac{n!}{(n-m)!} \right) \left(\frac{o!}{(o-m)!} \right) (o + n - m).$$

Thus, denoting n_i and o_i the number of individuals and objects remaining in the market at any round i, the random sequence (n_i, o_i) is a Markov chain.

[2]Sometimes, a tree is defined as an acyclic undirected connected graph. In such a case, a tree is rooted when we name one of its vertex a "root." Starting from such a rooted tree, if all edges now have a direction leading toward the root, then the out-degree of any vertex (except the root) is 1. So the two definitions are actually equivalent.

This theorem shows that the numbers of agents and objects that are assigned in each round of TTC follow a simple Markov chain depending *only* on the numbers of agents and objects at the beginning of that round. It also characterizes the probability structure of the Markov chain. This theorem implies that there are no conditioning issues at least with respect to the *total* numbers of agents and objects that are assigned in each round of TTC. Namely, one does not need to keep track of the precise history leading up to a particular economy at the beginning of a round, as far as the numbers of objects assigned in that round is concerned. Obviously, this result is crucial in rendering Lemma S2 analytically provable. However, this result alone is not sufficient. We still need to understand the number of objects that are assigned via short versus long cycles in each round. Unfortunately, the *composition* of cycles—long versus short—cleared in each round depends on the precise history leading up to the economy at the beginning of that round. The next two sections deal with this issue.

S.3.2 The Number of Objects Assigned via Short Cycles

We begin by noting that TTC induces a random sequence of spanning rooted forests. Indeed, one could see the beginning of the first round of TTC as a situation where we have the trivial forest consisting of $|I| + |O|$ trees with isolated vertices. Within this step each vertex in I will randomly point to a vertex in O and each vertex in O will randomly point to a vertex in I. Note that once we delete the realized cycles, we again get a spanning rooted forest. So we can think again of the beginning of the second round of TTC as a situation where we start with a spanning rooted forest where the agents and objects remaining from the first round form this spanning rooted forest, where the roots consist of those agents and objects that had pointed to the entities that were cleared via cycles. Here again objects that are roots randomly point to a remaining individual and individuals that are roots randomly point to a remaining object. Once cycles are cleared we again obtain a forest and the process goes on like this.

Formally, the random sequence of forests, $F_1, F_2, ...$ is defined as follows. First, we let F_1 be a trivial unique forest consisting of $|I|+|O|$ trees with isolated vertices, forming their own roots. For any $i = 2, ...$, we first create a random directed edge from each root of F_{i-1} to a vertex on the other side, and then delete the resulting cycles (these are the agents and objects assigned in round $i - 1$) and F_i is defined to be the resulting rooted forest.

We begin with the following question: If round k of TTC begins with a rooted forest F, what is the expected number of short-cycles that will form at the end of that round? We will show that, irrespective of F, this expectation is bounded by 2. To show this, we will make a couple of observations.

To begin, let n_k be the cardinality of the set I_k of individuals in our forest F and let o_k be the cardinality of O_k, the set of F's objects. And, let $A \subset I_k$ be the set of roots on the individuals side of our given forest F and let $B \subset O_k$ be the set of its roots on the objects side. Their cardinalities are a and b, respectively.

Now, observe that for any $(i, o) \in A \times B$, the probability that (i, o) forms a short-cycle is $\frac{1}{n_k}\frac{1}{o_k}$. For any $(i, o) \in (I_k \backslash A) \times B$, the probability that (i, o) forms a short-cycle is $\frac{1}{n_k}$ if i points to o and 0 otherwise. Similarly, for $(i, o) \in A \times (O_k \backslash B)$, the probability that (i, o) forms a short-cycle is $\frac{1}{o_k}$ if o points to i and 0 otherwise. Finally, for any $(i, o) \in (I_k \backslash A) \times (O_k \backslash B)$, the probability that (i, o) forms a short-cycle is 0 (by definition of a forest, i and o cannot be pointing to each other in the forest F). So, given the forest F, the expectation of the number S_k of short-cycles is

$$
\begin{aligned}
\mathbb{E}\left[S_k | F_k = F\right] &= \mathbb{E}\left[\sum_{(i,o)\in I_k \times O_k} \mathbf{1}_{\{(i,o) \text{ is a short-cycle}\}} \middle| F_k = F\right] \\
&= \sum_{(i,o)\in I_k \times O_k} \mathbb{E}\left[\mathbf{1}_{\{(i,o) \text{ is a short-cycle}\}} | F_k = F\right] \\
&= \sum_{(i,o)\in A \times B} \mathbb{E}\left[\mathbf{1}_{\{(i,o) \text{ is a short-cycle}\}} | F_k = F\right] \\
&\quad + \sum_{(i,o)\in (I_k \backslash A) \times B} \mathbb{E}\left[\mathbf{1}_{\{(i,o) \text{ is a short-cycle}\}} | F_k = F\right] \\
&\quad + \sum_{(i,o)\in A \times (O_k \backslash B)} \mathbb{E}\left[\mathbf{1}_{\{(i,o) \text{ is a short-cycle}\}} | F_k = F\right] \\
&= \sum_{(i,o)\in A \times B} \Pr\{(i, o) \text{ is a short-cycle} \,| F_k = F\} \\
&\quad + \sum_{(i,o)\in (I_k \backslash A) \times B} \Pr\{(i, o) \text{ is a short-cycle} \,| F_k = F\} \\
&\quad + \sum_{(i,o)\in I_k \times (O_k \backslash B)} \Pr\{(i, o) \text{ is a short-cycle} \,| F_k = F\} \\
&\leq \frac{ab}{n_k o_k} + \frac{n_k - a}{n_k} + \frac{o_k - b}{o_k} \\
&= 2 - \frac{a o_k + b n_k - ab}{n_k o_k} \leq 2.
\end{aligned}
$$

Observe that since $o_k \geq b$, the above term is smaller than 2. Thus, as claimed, we obtain the following result.[3]

[3]Note that the bound is pretty tight: if the forest F has one root on each side and each node which

Proposition S1. *If TTC round k begins with any forest F,*

$$\mathbb{E}\left[S_k \,|F_k = F\right] \leq 2.$$

Given that our upper bound holds for any forest F, we get the following corollary.

Corollary S2. *For any round k of TTC,* $\mathbb{E}\left[S_k\right] \leq 2$.

S.3.3 The Number of Objects Assigned via Long Cycles

Again consider the unbalanced market in which $|I| \geq |O|$, and recall $n := |I|$ and $o := |O|$.

The Markov property established in Theorem S1 means that the number of agents and objects assigned in any TTC round depends only on the number of agents and objects that round begins with, regardless of how many rounds preceded that round and what happened in those rounds. Hence, the distribution of the (random) number M_k of objects that would be assigned in any round of TTC that begins with n_k agents and $o_k(\leq n_k)$ objects is the same as that in the first round of TTC when there are n_k agents and $o_k(\leq n_k)$. In particular, we can apply Lemma S4 to compute its expected value:[4]

$$\mathbb{E}[M_k \mid |O_k| = o_k] = \sum_{i=1}^{o} \frac{(o_k)_i (n_k)_i}{o_k^i n_k^i}.$$

We can make two observations: First, the expected number is increasing in o_k (and n_k) and goes to infinity as o_k (and n_k) increases. This can be seen easily by the fact that $\frac{k-l}{k}$ is increasing in k for any $k > l$. Second, given our assumption that $n_k \geq o_k$, there exists $\hat{o} \geq 1$[5] such that

$$\mathbb{E}[M_k \,|o_k] \geq 3 \text{ if } o_k \geq \hat{o}.$$

We are now ready to present the main result. Recall that $\hat{O}$ is the (random) set of objects that are assigned via long cycles in TTC.

Theorem S2. $\mathbb{E}\left[\frac{|\hat{O}|}{|O|}\right] \geq \frac{1}{3} - \frac{\hat{o}-2}{3|O|}$.

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is not a root points to the (unique) root on the opposite side, the expected number of short-cycles given F is $\frac{1}{n_k o_k} + \frac{n_k - 1}{n_k} + \frac{o_k - 1}{o_k} \to 2$ as $n_k, o_k \to \infty$. Thus, the conditional expectation of s_k is bounded by 2 and, asymptotically, this bound is tight. However, we can show, using a more involved computation, that the unconditional expectation of s_k is bounded by 1. The details of the computation are available upon request.

[4]The number is half of that stated in Lemma S4 since the number of agents cleared in any round is precisely the half of the cyclic vertices in a random bipartite graph at the beginning of that round. Recall also that, by definition of TTC, together with our assumption that $o \leq n$, given the number of objects o_k, the number of individuals n_k is totally determined and is equal to $o_k + n - o$.

[5]One can check that $\hat{o} = 13$ works. In particular, if $n_k = o_k$, $\mathbb{E}[m_k \,|o_k] \geq 3$ if and only if $o_k \geq 13$.

Proof. Consider the following sequence of random variables $\{\mathbb{E}(L_k\,|o_k)\}_{k=1}^{|O|}$ where o_k is the number of remaining objects at round k while L_k is the number of objects assigned at round k via long cycles. (Note both are random variables.) Thus, $o_1 = |O|$. Note that $\mathbb{E}(L_{|O|}\,\big|o_{|O|}) = 0$. By Theorem S1, we are defining here the process $\{\mathbb{E}(L_k\,|o_k)\}_{k=1}^{|O|}$ induced by the Markov chain $\{o_k\}$. Note also that $\mathbb{E}(L_k\,|o_k) = \mathbb{E}[M_k\,|o_k] - \mathbb{E}[S_k\,|o_k]$ where S_k is the number of objects assigned at round k via short cycles. By Proposition S1, $\mathbb{E}[S_k\,|F] \leq 2$ for any possible forest F, this implies that $\mathbb{E}[S_k\,|o_k] \leq 2$. Hence, we obtain that $\mathbb{E}(L_k\,|o_k) \geq 3 - 2 = 1$ if $o_k \geq \hat{o}$. (Recall that $\hat{o}$ is defined such that $o_k \geq \hat{o}$ implies $\mathbb{E}[M_k\,|o_k] \geq 3$.) Let T be first round at which the $\mathbb{E}(L_k\,|o_k)$ becomes smaller than 1: formally, $\mathbb{E}(L_k\,|o_k) \leq 1$ only if $k \geq T$ (this is well-defined since $\mathbb{E}(L_{|O|}\,\big|o_{|O|}) = 0$). Note that $o_T \leq \hat{o}$.

Now we obtain:

$$\begin{aligned}
\mathbb{E}[|\hat{O}|] &= \mathbb{E}(\sum_{k=1}^{|O|} L_k) \\
&= \sum_{o\in O} \Pr\{\tilde{o}_k = o\} \sum_{k=1}^{|O|} \mathbb{E}\left[L_k \middle| \tilde{o}_k = o\right] \\
&= \sum_{t} \Pr\{T = t\} \sum_{o\in O} \Pr\{\tilde{o}_k = o \middle| T = t\} \sum_{k=1}^{|O|} \mathbb{E}\left[L_k \middle| \tilde{o}_k = o\right] \\
&\geq \sum_{t} \Pr\{T = t\} \sum_{k=1}^{t-1} \sum_{o\in O} \Pr\{\tilde{o}_k = o \middle| T = t\} \mathbb{E}\left[L_k \middle| \tilde{o}_k = o\right] \\
&\geq \sum_{t} \Pr\{T = t\}(t-1) \\
&= \mathbb{E}[T] - 1
\end{aligned}$$

where the last inequality holds by definition of the random variable T. Indeed, whenever $\Pr\{\tilde{o}_k = o \big| T = t\} > 0$ (recall that $k < t$), $\mathbb{E}\left[L_k \middle| \tilde{o}_k = o\right] \geq 1$ must hold.

Once we have reached round T under TTC, at most $\hat{o}$ more short cycles can arise. Thus, the expected number of short cycles must be smaller than $2\mathbb{E}(T) + \hat{o}$. Indeed, the expected number of short cycles is smaller than 2 times the expected number of rounds for TTC to converge (recall that, by Corollary S2, the expected number of short cycles at each round is at most two) which itself is smaller than $2\mathbb{E}(T) + \hat{o}$. It follows that

$$2\mathbb{E}[T] + \hat{o} \geq \mathbb{E}[|O| - |\hat{O}|].$$

Combining the above inequalities, we obtain that

$$\mathbb{E}[|\hat{O}|] \geq \frac{1}{3}\left(|O| - \hat{o} + 2\right),$$

from which the result follows. ∎

Lemma S2 is a direct corollary of Theorem S2.

S.4 Proof of Theorem 3

Since $U(u_1^0, 0) > U(u_2^0, 1))$, all objects in O_1 are assigned before any agent starts applying to objects in O_2. Hence, the assignment achieved by individuals assigned objects in O_1 is the same as the one obtained when we run DA in the submarket with individuals in I and objects in O_1. The following lemma shows that the agents assigned objects in O_1 suffer a significant number of rejections before getting assigned. This result is obtained by Ashlagi, Kanoria, and Leshno (2017) and by Ashlagi, Braverman, and Hassidim (2011). We provide a much simpler direct proof for this result here.[6]

Lemma S5 (Welfare Loss under Unbalanced Market)**.** *Consider an unbalanced submarket consisting of agents I and objects O_1, where $|I| - |O_1| \to n(1 - x_1)$ as $n \to \infty$. Let I_1 be the (random) set of agents who are assigned objects in O_1, and let $I_1^\delta := \{i \in I_1 | i$ makes at least δn offers$\}$ be the subset of them who each suffer from more than δn rejections (before getting assigned objects in O_1). Then, there exist γ, δ, υ, all strictly positive, such that for all $n > N$ for some $N > 0$,*

$$\Pr\left\{ \frac{|I_1^\delta|}{|I|} > \gamma \right\} > \upsilon.$$

Proof. Without loss, we work with the McVitie and Wilson's algorithm (which equivalently implement DA). Consider the individual $i = n$ at the last serial order, at the beginning of step n. By that step, each object in O_1 has surely received at least $|I| - |O_1|$ offers. This is because at least $|I| - |O_1| - 1$ preceding agents must be unassigned, so each of them must have been rejected by all objects in O_1 before the beginning of step n.

Each object receives offers randomly and selects its most preferred individual among those who have made offers to that object. Since each object will have received at least $|I| - |O_1|$ offers, its payoff must be at least $\max\{\eta_{1,o}, ...\eta_{|I|-|O_1|,o}\}$, i.e., the maximum of $|I| - |O_1|$ random draws of its idiosyncratic payoffs. At the beginning of step n, agent n makes an offer to an object o (i.e., his most favorite object which is drawn iid). Then, for n to be accepted by o, it must be the case that $\eta_{i,o} \geq \max\{\eta_{1,o}, ...\eta_{|I|-|O_1|,o}\}$. This occurs with probability $\frac{1}{|I|-|O_1|}$. Thus, the probability that n is assigned o is at most $\frac{1}{|I|-|O_1|}$.

[6]The main case studied by Ashlagi, Kanoria, and Leshno (2017) deals with the situation in which the degree of unbalancedness is small; i.e., $|I| - |O_1|$ is sublinear in n. Our proof does not apply to that case.

Hence, for any $\delta \in (0, x_1)$, the probability that agent n is rejected δn times in a row is at least

$$\left(1 - \frac{1}{|I| - |O_1|}\right)^{\delta n} \to \left(\frac{1}{e}\right)^{\frac{\delta}{1-x_1}}.$$

Since agent n is ex ante symmetric with all other agents, for any agent $i \in I$,

$$\liminf \Pr\left\{\mathcal{E}_i^\delta\right\} \geq \left(\frac{1}{e}\right)^{\frac{\delta}{1-x_1}},$$

for any $\delta \in (0, x_1)$, where $\mathcal{E}_i^\delta$ denotes the event that i makes at least δn offers.

Let $\mathcal{F}_i := \{i \in I_1\}$ denote the event that agent i is assigned an object in O_1, and let $\mathcal{F}_i^c := \{i \notin I_1\}$ be its complementary event. Then, by ex ante symmetry of all agents, $\Pr\{\mathcal{F}_i\} = |O_1|/n \to x_1$ as $n \to \infty$. For $\delta \in (0, x_1)$, we obtain

$$\begin{aligned}\Pr\left\{\mathcal{E}_i^\delta\right\} &= \Pr\{\mathcal{F}_i\} \Pr\left\{\mathcal{E}_i^\delta \mid \mathcal{F}_i\right\} + \Pr\{\mathcal{F}_i^c\} \Pr\left\{\mathcal{E}_i^\delta \mid \mathcal{F}_i^c\right\} \\ &\to x_1 \Pr\left\{\mathcal{E}_i^\delta \mid \mathcal{F}_i\right\} + (1 - x_1) \cdot 1 \text{ as } n \to \infty,\end{aligned}$$

where the last line obtains since, with probability going to one as $n \to 1$, an agent who is not assigned an object in O_1 must make at least $\delta n < x_1 n$ offers. Combining the two facts, we have

$$\liminf \Pr\left\{\mathcal{E}_i^\delta \mid \mathcal{F}_i\right\} \geq \frac{1}{x_1}\left(\left(\frac{1}{e}\right)^{\frac{\delta}{1-x_1}} - (1 - x_1)\right).$$

Observe that the RHS tends to a strictly positive number as $\delta \to 0$. Thus, for $\delta > 0$ small enough (smaller than $(1 - x_1)\log(\frac{1}{x_1})$), $\Pr\left\{\mathcal{E}_i^\delta \mid \mathcal{F}_i\right\}$ is bounded below by some positive constant for all n large enough.

It thus follows that there exist $\delta \in (0, x_1)$, $\gamma > 0$ such that

$$\begin{aligned}\mathbb{E}\left[\frac{|I_1^\delta|}{|I|}\right] &= \frac{1}{|I|}\mathbb{E}\left[\sum_{i \in I_1} \mathbf{1}_{\mathcal{E}_i^\delta}\right] \\ &= \frac{1}{|I|}\mathbb{E}_{I_1}\mathbb{E}\left[\sum_{i \in I_1} \mathbf{1}_{\mathcal{E}_i^\delta} \middle| I_1\right] \\ &= \frac{|I_1|}{|I|}\mathbb{E}_{I_1}\mathbb{E}\left[\mathbf{1}_{\mathcal{E}_i^\delta} | i \in I_1\right] \\ &= \frac{|I_1|}{|I|}\mathbb{E}\left[\mathbf{1}_{\mathcal{E}_i^\delta} | i \in I_1\right] \\ &= \frac{|I_1|}{|I|} \Pr\left\{\mathcal{E}_i^\delta | \mathcal{F}_i\right\} \\ &> \gamma\end{aligned}$$

for all $n > N$ for some $N > 0$, since $\frac{|I_1|}{|I|} \to x_1$ as $n \to \infty$. The claimed result then follows. ∎

Lemma S5 implies that there exists $\epsilon' > 0, \upsilon' > 0, \gamma' > 0$ such that for all $n > N'$ for some $N' > 0$,

$$\Pr\left\{\frac{|\tilde{I}_\epsilon|}{|I|} \geq \gamma'\right\} \geq \upsilon',$$

where $\tilde{I}_{\epsilon'} := \{i \in I | DA(i) \in O_1, U_i(DA(i)) \leq U(u_1, 1-\epsilon')\}$ is the set of agents assigned to objects in O_1 but receive payoffs bounded above by $U(u_1, 1-\epsilon')$.

Now consider a matching mechanism μ that first runs DA and then runs a Shapley-Scarf TTC afterwards, namely the TTC with the DA assignments serving as the initial endowments for the agents. This mechanism μ clearly Pareto dominates DA. In particular, if $DA(i) \in O_1$, then $\mu(i) \in O_1$. For any ϵ'', let

$$\check{I}_{\epsilon''} := \{i \in I | \mu(i) \in O_1, U_i(DA(i)) \geq U(u_1, 1-\epsilon'')\},$$

be those agents who attain at least $U(u_1, 1-\epsilon'')$ from μ. By Lemma 1, we have for any ϵ'', γ'' and υ'', such that

$$\Pr\left\{\frac{|\check{I}_{\epsilon''}|}{|I|} \leq \gamma''\right\} < \upsilon'',$$

for all $n > N''$ for some $N'' > 0$.

Now set ϵ', ϵ'' such that $\epsilon = \epsilon' - \epsilon'' > 0$, γ', γ'' such that $\gamma := \gamma' - \gamma'' > 0$, and υ', υ'' such that $\upsilon := \upsilon' - \upsilon'' > 0$. Observe that $I_\epsilon(\mu|DA) \supset \tilde{I}_{\epsilon'} \setminus \check{I}_{\epsilon''}$, so $|I_\epsilon(\mu|DA)| \geq |\tilde{I}_{\epsilon'}| - |\check{I}_{\epsilon''}|$. It then follows that for all $n > N := \max\{N', N''\}$,

$$\begin{aligned}
\Pr\left\{\frac{|I_\epsilon(\mu|DA)|}{|I|} \geq \gamma\right\} &\geq \Pr\left\{\frac{|\tilde{I}_{\epsilon'}|}{|I|} - \frac{|\check{I}_{\epsilon''}|}{|I|} \geq \gamma\right\} \\
&\geq \Pr\left\{\frac{|\tilde{I}_{\epsilon'}|}{|I|} \geq \gamma' \text{ and } \frac{|\check{I}_{\epsilon''}|}{|I|} \leq \gamma''\right\} \\
&\geq \Pr\left\{\frac{|\tilde{I}_{\epsilon'}|}{|I|} \geq \gamma'\right\} - \Pr\left\{\frac{|\check{I}_{\epsilon''}|}{|I|} > \gamma''\right\} \\
&\geq \upsilon' - \upsilon = \upsilon.
\end{aligned}$$

S.5 The Erdös-Renyi mechanim

We first briefly recall the Erdös-Renyi theorem mentioned in the paper. It is thus worth introducing the relevant model of random graph. A **bipartite graph** G consists in vertices, $V_1 \cup V_2$, and edges $E \subset V_1 \times V_2$ across V_1 and V_2 (with no possible edges within vertices in each side). A **perfect bipartite matching** is a bipartite graph in which each vertex is involved in exactly one edge. A random bipartite graph $B = (V_1 \cup V_2, p)$, $p \in (0, 1)$, is a bipartite graph with vertices $V_1 \cup V_2$ in which each pair $(v_1, v_2) \in V_1 \times V_2$ is linked by an edge with probability p independently (of edges created for all other pairs). In this context, the Erdös-Renyi theorem can be stated as follows.

Theorem S3. *Consider a random bipartite graph $B = (V_1 \cup V_2, p)$ where $0 < p < 1$ is a constant and for each $i \in \{1, 2\}$ and $|V_1| = |V_2| = n$. Then, the probability that the random graph admits a perfect bipartite graph—i.e., the probability that one can construct a perfect bipartite graph using a subset of edges—goes to one as $n \to \infty$.*

Hence, in our environment where we draw randomly individuals' preferences and objects' priorities, one can build an associated random bipartite graph which consists of vertices $I \cup O$ and where an edge between (i, o) is added if and only if $\xi_{i,o} > 1 - \epsilon$ and $\eta_{i,o} > 1 - \epsilon$. Applying the Erdös-Renyi theorem we obtain that with probability approaching 1 as $|I| = |O| = n$ increases, a (perfect bipartite) matching exists where all objects and agents realize idiosyncratic payoffs greater than $1 - \epsilon$.[7] Hence, one could construct a mechanism in which (1) agents and objects (more precisely their suppliers) report their idiosyncratic shocks (2) given such a reports a bipartite graph is built where an edge between an agent and an object if and only if $\xi_{i,o} > 1 - \epsilon$ and $\eta_{i,o} > 1 - \epsilon$ and (3) a maximal bipartite matching is selected.[8] Such mechanism would select a perfect matching whenever it exists. Under truthful reports, this occurs with probability approaching 1 as the market size increases. Thus, by construction, this mechanism is asymptotically efficient and asymptotically stable. In addition, well-known polynomials algorithms such as the *augmenting path algorithm* would find a maximal matching.

However, as stated in the paper, this mechanism would not be desirable for several reasons. First, it would not work if the agents cannot tell apart common values from idiosyncratic values. More importantly, the mechanism would not have a good incentive property. An agent will be reluctant to report the objects in lower tiers even though they have high idiosyncratic preferences. Indeed, if he expects that with significant probability, he will not get any object in the highest tier, he will have incentives to claim that he enjoys high idiosyncratic payoffs with a large number of high tier objects and that all his idiosyncratic payoffs for the other tiers are low. It is very likely that there is a perfect matching even under this misreport and this will ensure him to get matched with a high tier object.

[7] The unbalanced case can be treated with almost no modifications.

[8] A maximal bipartite matching is a matching that contains the largest possible number of edges.

S.6 Completion of the Proof of Theorem 4

We start by showing that DACB with $\kappa \geq \log^2(n)$ and $\kappa = o(n)$ is asymptotically efficient. Let $\mu :=$ DACB, we have to show that for any $\epsilon > 0$, and any mechanism μ' that weakly Pareto-dominates μ

$$\frac{I_\epsilon(\mu' \,|\mu)}{n} \xrightarrow{p} 0$$

where we recall that $I_\epsilon(\mu' \,|\mu) := \{i \in I \,| U(\mu(i)) < U(\mu'(i)) - \epsilon\}$.

We first introduce some notations: In the sequel, for an arbitrary collection of sets $\{X_k\}_{k=1}^K$, we let $X_{<k}$ (resp. $X_{\leq k}$) be the set $\cup_{\ell<k} X_k$ (resp. $\cup_{\ell \leq k} X_k$). Note that $X_{<1} = \emptyset$. We fix any mechanism μ' that weakly Pareto-dominates μ and any $\epsilon > 0$ small enough so that $U(u_k, 1) - \epsilon > U(u_{k+1}, 1)$ for all $k = 1, ..., K-1$.

Now, fix $k \geq 1$ and let $I'_k := \{i \in I | \mu(i) \in O_k\}$ be the set of individuals matched to objects in O_k under matching μ' and recall that I_k is the set of individuals matched to objects in O_k under matching μ. By definition, $|I'_{<k}| = |I_{<k}|$. Let

$$\bar{I}_k := \{i \in I_k \,| U(\mu(i)) \geq U(u_k, 1) - \epsilon\} \text{ and } \hat{I}_{\epsilon,k}(\mu' \,|\mu) := \{i \in I_k \,| U(\mu(i)) \geq U(\mu'(i)) - \epsilon\}.$$

Then, for any $i \in \bar{I}_k \backslash I'_{<k}$, $U(\mu(i)) \geq U(u_k, 1) - \epsilon \geq U(\mu'(i)) - \epsilon$. Hence, we must have $\bar{I}_k \backslash I'_{<k} \subset \hat{I}_{\epsilon,k}(\mu' \,|\mu)$. Observe next $\bar{I}_{<k} \subset I'_{<k}$. This follows since, if $i \in \bar{I}_{<k}$ (note that this implies $k \geq 2$), then by definition $U(\mu'(i)) \geq U(\mu(i)) \geq U(u_{k-1}, 1) - \epsilon > U(u_k, 1)$ and so $i \in I'_{<k}$.

We fix $\gamma' < 1$ and show that, as $n \to \infty$,

$$\Pr\left\{\frac{\left|\hat{I}_{\epsilon,k}(\mu' \,|\mu)\right|}{|I_k|} \geq \gamma'\right\} \to 1.$$

For $k = 1$, as $n \to \infty$,

$$\Pr\left\{\frac{\left|\hat{I}_{\epsilon,1}(\mu' \,|\mu)\right|}{|I_1|} \geq \gamma'\right\} \geq \Pr\left\{\frac{\left|\bar{I}_1 \backslash I'_{<1}\right|}{|I_1|} \geq \gamma'\right\} = \Pr\left\{\frac{\left|\bar{I}_1\right|}{|I_1|} \geq \gamma'\right\} \to 1,$$

by Proposition 1. Next consider $k \geq 2$ and fix any $\delta > 0$ such that $\gamma' + \delta < 1$. Then, as

$n \to \infty$,

$$\begin{aligned}
\Pr\left\{\frac{\left|\hat{I}_{\epsilon,k}(\mu'\,|\mu)\right|}{|I_k|} \geq \gamma'\right\} &\geq \Pr\left\{\frac{\left|\bar{I}_k \backslash I'_{<k}\right|}{|I_k|} \geq \gamma'\right\} \\
&= \Pr\left\{\frac{\left|\bar{I}_k \backslash \left((I'_{<k} \cap \bar{I}_{<k}) \cup (I'_{<k} \backslash \bar{I}_{<k})\right)\right|}{|I_k|} \geq \gamma'\right\} \\
&\geq \Pr\left\{\frac{\left|\bar{I}_k \backslash (I'_{<k} \cap \bar{I}_{<k})\right| - \left|I'_{<k} \backslash \bar{I}_{<k}\right|}{|I_k|} \geq \gamma'\right\} \\
&= \Pr\left\{\frac{\left|\bar{I}_k\right|}{|I_k|} - \frac{\left|I'_{<k} \backslash \bar{I}_{<k}\right|}{|I_k|} \geq \gamma'\right\} \\
&= \Pr\left\{\frac{\left|\bar{I}_k\right|}{|I_k|} - \left(\frac{|I'_{<k}|}{|I_k|} - \frac{\left|\bar{I}_{<k}\right|}{|I_k|}\right) \geq \gamma'\right\} \\
&\geq \Pr\left\{\frac{\left|\bar{I}_k\right|}{|I_k|} \geq \gamma' + \delta \text{ and } \left(\frac{|I'_{<k}|}{|I_k|} - \frac{\left|\bar{I}_{<k}\right|}{|I_k|}\right) \leq \delta\right\} \\
&= \Pr\left\{\frac{\left|\bar{I}_k\right|}{|I_k|} \geq \gamma' + \delta \text{ and } \frac{\left|\bar{I}_{<k}\right|}{|I_{<k}|} \geq 1 - \delta\frac{|I_k|}{|I_{<k}|}\right\} \to 1,
\end{aligned}$$

where the second equality holds since $\bar{I}_k \cap \bar{I}_{<k} = \emptyset$, which implies $\bar{I}_k \backslash (I'_{<k} \cap \bar{I}_{<k}) = \bar{I}_k$; the third equality holds since $\bar{I}_{<k} \subset I'_{<k}$; the last equality uses $|I'_{<k}| = |I_{<k}|$; and the convergence result follows from Proposition 1.

To complete the argument, fix any $\gamma > 0$. The, the desired result holds since

$$\begin{aligned}
\Pr\left\{\frac{|I_\epsilon(\mu'\,|\mu)|}{n} \geq \gamma\right\} &= \Pr\left\{\sum_{k=1}^{K}\frac{\left|\hat{I}_{\epsilon,k}(\mu'\,|\mu)\right|}{n} < 1 - \gamma\right\} \\
&\leq \Pr\left\{\frac{\left|\hat{I}_{\epsilon,k}(\mu'\,|\mu)\right|}{n} < \frac{1-\gamma}{K} \text{ for some } k\right\} \\
&\leq \sum_{k=1}^{K}\Pr\left\{\frac{\left|\hat{I}_{\epsilon,k}(\mu'\,|\mu)\right|}{n} < \frac{1-\gamma}{K}\right\} \\
&= \sum_{k=1}^{K}\left(1 - \Pr\left\{\frac{\left|\hat{I}_{\epsilon,k}(\mu'\,|\mu)\right|}{n} \geq \frac{1-\gamma}{K}\right\}\right) \to 0
\end{aligned}$$

as $n \to \infty$, where the second inequality uses the union bound, and the convergence follows from the above arguments (with $\gamma' := \frac{1-\gamma}{K}$).

We now prove that DACB with $\kappa \geq \log^2(n)$ and $\kappa = o(n)$ is asymptotically stable. We must show that

$$\frac{|J_\epsilon(\mu)|}{n(n-1)} \xrightarrow{p} 0,$$

where we recall that $J_\epsilon(\mu) := \{(i,o) \in I \times O \,|U_i(o) > U_i(\mu(i)) + \epsilon \text{ and } V_o(i) > V_o(\mu(o)) + \epsilon\}$. Fix any $\gamma > 0$, we have

$$\begin{aligned}
\Pr\left\{\frac{|J_\epsilon(\mu)|}{n(n-1)} \geq \gamma\right\} &\leq \Pr\left\{\frac{|I \times \{o \in O \,|V(1) > V_o(\mu(o)) + \epsilon\}|}{n(n-1)} \geq \gamma\right\} \\
&= \Pr\left\{\frac{|\{o \in O \,|V(1) > V_o(\mu(o)) + \epsilon\}|}{n-1} \geq \gamma\right\} \\
&= \Pr\left\{\frac{1}{n-1}\sum_{k=1}^{K} |\{o \in O_k \,|V(1) > V_o(\mu(o)) + \epsilon\}| \geq \gamma\right\} \\
&\leq \Pr\left\{\frac{1}{n-1} |\{o \in O_k \,|V(1) > V_o(\mu(o)) + \epsilon\}| \geq \frac{\gamma}{K} \text{ for some } k\right\} \\
&\leq \sum_{k=1}^{K} \Pr\left\{\frac{1}{n-1} |\{o \in O_k \,|V(1) > V_o(\mu(o)) + \epsilon\}| \geq \frac{\gamma}{K}\right\} \\
&= \sum_{k=1}^{K} \Pr\left\{\frac{|\{o \in O_k \,|V(1) > V_o(\mu(o)) + \epsilon\}|}{|O_k|} \geq \frac{\gamma}{K}\frac{n-1}{|O_k|}\right\} \to 0
\end{aligned}$$

as $n \to \infty$, where the last inequality is by the union bound while the convergence result holds by Proposition 1.

S.7 Convergence Rates for DACB

In the sequel, we consider DACB where $j(n) = 1$ and $5\log^2(n) \leq \kappa(n) = o(n)$. Fix any $\varepsilon > 0$, $\delta > 0$ and μ' which Pareto-dominates $\mu :=$ DACB. We want to show that

$$\Pr\left\{\frac{|I_\varepsilon(\mu' \,|\mu)|}{n} < \delta\right\}$$

converges to 1 at a rate greater than $1/n$. (Formally, we show that for all $\gamma > 0$, there exists N such that for any $n > N$, $\Pr\left\{\frac{|I_\varepsilon(\mu'|\mu)|}{n} < \delta\right\} \geq \gamma\left(1 - \frac{1}{n}\right)$ which is usually denoted by $\Pr\left\{\frac{|I_\varepsilon(\mu'|\mu)|}{n} < \delta\right\} = \omega\left(1 - \frac{1}{n}\right)$.)

We will actually show a stronger result:

$$\Pr\left\{\frac{|I_\varepsilon(\mu' \,|\mu)|}{n} = 0\right\}$$

converges to 1 at an order greater than $1/n$. Given our arguments in the paper, it is enough to show that

$$\Pr\left\{\forall i \in I : \xi_{i,\mu(i)} \geq 1-\varepsilon\right\}$$

converges to 1 at an order greater than $1/n$. Because of this, note that the rate of convergence does not depend on the specific choice of μ'.

We know from the proof of Proposition 1 that $\left\{\forall i \in I : \xi_{i,\mu(i)} \geq 1-\varepsilon\right\}$ holds whenever for each $k = 1, ..., K$, the following two events are satisfied (1) E_1^k : "all agents' $\kappa(n)$ favorite objects in O_k yield an idiosyncratic payoff higher than $1-\varepsilon$" and (2) E_2^k : "all objects in tier k are assigned in Stage k". By the proof of Lemma S1-(i), we know there is $\rho > 0$, so that for n large enough (i.e., such that $\kappa(n) \leq \rho n$),

$$\Pr(E_1^k) \geq 1 - \Pr\{\exists i \in I : X_i \leq \rho\,|O_k|\} \geq 1 - \frac{n}{\exp(cn)}$$

where $c > 0$ and X_i is the number of objects in O_k with which i enjoys an idiosyncratic payoff higher than $1-\varepsilon$.

Now, to compute $\Pr(E_2^k)$, recall that E_2^k holds whenever the following two events hold: F_1^k : "all agents' $\kappa(n)$ favorite objects in $O_{\geq k}$ are in O_k" and F_2^k : "when at the beginning of Stage k, we restrict our attention to the submarket composed of the agents (among remaining ones) with the $|O_k|$ lowest serial orders and to objects in O_k then DA converges before an agent has made more than $\kappa(n)$ offers". In the sequel, without loss of generality, we assume that ε is small enough so that $u_k + 1 - \varepsilon > u_{k+1} + 1$ for each $k = 1, ..., K-1$. By our choice of $\varepsilon > 0$, we know that $\Pr(F_1^k) \geq \Pr(E_1^k) \geq 1 - \frac{n}{\exp(cn)}$. Now, by Pittel (1992) we know that whenever $\kappa(n) \geq 5\log^2(n)$,

$$\Pr(F_2^k) \geq 1 - O(\frac{1}{n^d})$$

where $d > 1$.

From the above, we obtain

$$
\begin{aligned}
\Pr\left\{\forall i\in I:\xi_{i,\mu(i)}\geq 1-\varepsilon\right\} &\geq \Pr\left(\bigcap_{k=1}^{K}(E_1^k\cap E_2^k)\right)\\
&\geq \Pr\left(\bigcap_{k=1}^{K}(E_1^k\cap F_1^k\cap F_2^k)\right)\\
&\geq \sum_{k=1}^{K}\Pr(E_1^k\cap F_1^k\cap F_2^k)-(K-1)\\
&\geq \sum_{k=1}^{K}\left(\Pr(E_1^k)+\Pr(F_1^k)+\Pr(F_2^k)-2\right)-(K-1)\\
&\geq \sum_{k=1}^{K}\left(1-\frac{n}{\exp(cn)}+1-\frac{n}{\exp(cn)}+1-O(\frac{1}{n^d})-2\right)-(K-1)\\
&= 1-2K\frac{n}{\exp(cn)}-KO(\frac{1}{n^d})=1-O(\frac{1}{n^d})
\end{aligned}
$$

where the third and fourth inequality come from iterative applications of the rule $\Pr(A\cap B)\geq\Pr(A)+\Pr(B)-1$. Since $d>1$, we obtain the desired result.

Let us move to a similar exercice but now for objects. Fix any $\varepsilon>0$, $\delta>0$ and let $\mu:=$ DACB. We want to show that

$$\Pr\left\{\frac{|J_\varepsilon(\mu)|}{n(n-1)}<\delta\right\}$$

converges to 1 at a rate greater than $1/n$. Fix any function $f(n)=o(n)$ which satisfies $f(n)/\left(n/\log(n)\right)\to\infty$.

We start by showing that for n large enough and for each $k=1,...,K$:

$$\Pr\left\{\frac{|\{o\in O_k\,|V(\mu(o))\geq V(1)-\varepsilon\}|}{n-1}>\frac{\delta}{K}\right\}\geq 1-O(n^{-d})$$

where $d>1$. We prove it below for $k=1$, the proof for the other tiers is similar.

We know by the argument in the proof of Proposition 1 that whenever the event F_1^1 : "all agents' $\kappa(n)$ favorite objects in O are in O_1" holds then as long as agents make less than $\kappa(n)$ offers in Stage 1, DACB is the same as DA in the submarket composed only of agents with a serial order below $|O_1|$ and of all objects in O_1. Again, by Pittel (1992), because $\kappa(n)\geq 5\log^2(n)\geq 5\log^2(|O_1|)$ with probability greater than $1-O(n^{-d_1})$ where $d_1>1$, DA assigns all objects in O_1 before any agent makes more than $\kappa(n)$ offers. Hence, with probability greater than $1-O(n^{-d_1})$, the $|O_1|$ first steps of DACB are exactly the same as those in DA in the submarket. In addition, we know that under DA in this submarket:

$$\Pr\left\{\frac{1}{|O_1|}\sum_{o\in O_1} R_o^{DA} \le 2\frac{|O_1|}{\log(|O_1|)}\right\} \ge 1-O(n^{-d_2})$$

where $d_2 > 1$ and where the inequality holds by the main proposition in Pittel (1989). Now, for n large enough we have

$$\Pr\left\{\frac{\left|\left\{o\in O_1 \,\middle|\, R_o^{DA}\le f(n)\right\}\right|}{|O_1|} \ge 1-\frac{\delta}{K}\right\} \ge \Pr\left\{\frac{1}{|O_1|}\sum_{o\in O_1} R_o^{DA} \le 2\frac{|O_1|}{\log(|O_1|)}\right\} \ge 1-O(n^{-d_2})$$

where the first inequality holds because $f(n)/\left(n/\log(n)\right) \to \infty$.[9] We obtain that (conditional on F_1^1) the joint event $\left\{\frac{\left|\left\{o\in O_1 \middle| R_o^{DA}\le f(n)\right\}\right|}{|O_1|} \ge 1-\frac{\delta}{K}\right\}$ and "the $|O_1|$ first steps of DACB are exactly the same as those in DA in the submarket" occurs with probability greater than $1-O(n^{-d_1})-O(n^{-d_2}) = 1-O(n^{-d})$ where $d>1$. Finally, if for each $o\in O_1$, we denote R_o^{DACB} the rank of the match obtained by object o under DACB, we must have that conditional on F_1^1 occurring,

$$\Pr\left\{\frac{\left|\left\{o\in O_k \,\middle|\, R_o^{DACB}\le f(n)\right\}\right|}{|O_k|} \ge 1-\frac{\delta}{K}\right\} \ge 1-O(n^{-d}).$$

[9] For a contradiction, suppose there exists a sequence of $\{n_k\}_{k=1}^{\infty}$ such that $n_k\to\infty$ satisfying

$$\Pr\left\{\frac{\left|\left\{o\in O_1 \,\middle|\, R_o^{DA}\le f(n_k)\right\}\right|}{|O_1|} \ge 1-\frac{\delta}{K}\right\} < \Pr\left\{\frac{1}{|O_1|}\sum_{o\in O_1} R_o^{DA} \le 2\frac{|O_1|}{\log(|O_1|)}\right\}.$$

Then, we would have

$$\Pr\left\{\frac{\left|\left\{o\in O_1 \,\middle|\, R_o^{DA}> f(n_k)\right\}\right|}{|O_1|} > \frac{\delta}{K}\right\} > \Pr\left\{\frac{1}{|O_1|}\sum_{o\in O_1} R_o^{DA} > 2\frac{|O_1|}{\log(|O_1|)}\right\}. \tag{S0}$$

Next, observe $\frac{\left|\left\{o\in O_1 \middle| R_o^{DA}> f(n_k)\right\}\right|}{|O_1|} > \frac{\delta}{K} \implies \frac{1}{|O_1|}\sum_{o\in O_1} R_o^{DA} > \frac{\delta}{K} f(n_k)$. Hence, for n_k large enough (using the fact that $f(n)/\left(n/\log(n)\right)\to\infty$ implies $\frac{\delta}{K}f(n) > 2\frac{|O_1|}{\log(|O_1|)}$), we must have

$$\Pr\left\{\frac{\left|\left\{o\in O_1 \,\middle|\, R_o^{DA}> f(n_k)\right\}\right|}{|O_1|} > \frac{\delta}{K}\right\} \le \Pr\left\{\frac{1}{|O_1|}\sum_{o\in O_1} R_o^{DA} > 2\frac{|O_1|}{\log(|O_1|)}\right\}.$$

This contradicts (S0).

Denote by H^1 the event $\left\{\frac{\left|\left\{o\in O_1 \middle| R_o^{DACB}\le f(n)\right\}\right|}{|O_1|}\ge 1-\frac{\delta}{K}\right\}$. We obtain that

$$\begin{aligned}\Pr(H^1) &\geq \Pr(H^1 \,|F_1^1\,) \Pr(F_1^1) \\ &\geq \left(1-O(n^{-d})\right)\left(1-\frac{n}{\exp(cn)}\right)=1-O(n^{-d}).\end{aligned}$$

By the proof of Lemma S1-(iii), there must exist $\rho>0$ small enough so that, with probability greater than $1-\frac{n}{\exp(cn)}$, for each object $o\in O_1$, the ρn individuals with the highest priority yield o a payoff higher than $V(1)-\varepsilon$. Now, for n large enough, we have $f(n)/n\le\rho$ (since $f(n)=o(n)$). Hence, for n large enough, with probability greater than $1-\frac{n}{\exp(cn)}$, for each object in O_1, the $f(n)$ individuals with the highest priority yield o a payoff higher than $V(1)-\varepsilon$ (call this event G^1). Hence, we obtain that for n large enough

$$\begin{aligned}\Pr\left\{\frac{|\{o\in O_1\,|V(\mu(o))\ge V(1)-\varepsilon\}|}{n-1}\ge 1-\frac{\delta}{K}\right\} &\geq \Pr\left(G^1\cap H^1\right) \\ &\geq \Pr\left(G^1\right)+\Pr\left(H^1\right)-1 \\ &\geq 1-\frac{n}{\exp(cn)}-O(n^{-d})=1-O(n^{-d}),\end{aligned}$$

as was to be shown.

Now, we can complete the proof as follows

$$\begin{aligned}\Pr\left\{\frac{|J_\varepsilon(\mu)|}{n(n-1)}<\delta\right\} &\geq \Pr\left\{\sum_{k=1}^{K}\frac{|I\times\{o\in O_k\,|V(\mu(o))\le V(1)-\varepsilon\}|}{n(n-1)}<\delta\right\} \\ &\geq \Pr\left\{\forall k=1,...,K:\frac{|\{o\in O_k\,|V(\mu(o))\le V(1)-\varepsilon\}|}{n-1}<\frac{\delta}{K}\right\} \\ &= 1-\Pr\left\{\exists k=1,...,K:\frac{|\{o\in O_k\,|V(\mu(o))\le V(1)-\varepsilon\}|}{n-1}\ge\frac{\delta}{K}\right\} \\ &\geq 1-K\Pr\left\{\frac{|\{o\in O_k\,|V(\mu(o))\le V(1)-\varepsilon\}|}{n-1}\ge\frac{\delta}{K}\right\} \\ &= 1-K\left(1-\Pr\left\{\frac{|\{o\in O_k\,|V(\mu(o))\ge V(1)-\varepsilon\}|}{n-1}\ge 1-\frac{\delta}{K}\right\}\right) \\ &= 1-O(n^{-d}).\end{aligned}$$

S.8 Proof of Theorem 5

Let $f: I\to\{1,\ldots,n\}$ denote the serial orders for the agents. Recall by the basic uncertainty assumption, the distribution of the serial orders is such that for each agent i and any $\ell=1,\ldots,n$, $\Pr\{f(i)=\ell\}$ goes to 0 as n goes to infinity.

Let us fix $\varepsilon > 0$ and $k = 1, ..., K$. Assume that f gives to agent i a serial order in $\{|O_{\leq k-1}|+2, ..., |O_{\leq k}|\}$ with the convention that $|O_{\leq 0}|+2 = 1$. We show that there is $N \geq 1$ such that for any $n \geq N$, for any vector of cardinal utilities $(\hat{u}_o)_{o\in O} := (U_i(u_o, \xi_{io}))_{o\in O}$, i cannot gain more than ε by deviating given that everyone else reports truthfully. As will be clear, the argument does not depend on the specific serial order of i within $\{|O_{\leq k-1}| + 2, ..., |O_{\leq k}|\}$ and since there are finitely many tiers, N can be taken to be uniform across all individuals with serial order in $\cup_{k=1}^{K}\{|O_{\leq k-1}| + 2, ..., |O_{\leq k}|\}$. Hence, conditional on the event that i's serial order is in $\{|O_{\leq k-1}| + 2, ..., |O_{\leq k}|\}$ for some $k = 1, ..., K$, it will follow that for any $n \geq N$, for any vector of cardinal utilities, i cannot gain more than ε by deviating given that everyone else reports truthfully. Now, by assumption, the probability of the event that i's serial order is in $\{|O_{\leq k-1}| + 2, ..., |O_{\leq k}|\}$ for some $k = 1, ..., K$ tends to 1 as n goes to infinity and so even without conditioning, i cannot gain more than ε by deviating given that everyone else reports truthfully.

Before starting the proof of Theorem 5, we state the following lemma.

Lemma S6. *Let us assume that $\kappa(n) \geq \log^2(n)$ and $\kappa(n) = o(n)$ and consider DACB mechanism where all agents report truthfully. Fix any $k = 1, \ldots, K$ and any agent i with a serial order in $\{|O_{\leq k-1}|+2, ..., |O_{\leq k}|\}$. Assuming all agents report truthfully, the probability that i is matched at Stage k converges to 1 as n goes to infinity.*

Proof. By the argument in the proof of Proposition 1, we know that with probability approaching 1 as n goes to infinity, Step $|O_{\leq k}|$ ends under DACB and, for any agent i with a serial order in $\{|O_{\leq k-1}| + 2, ..., |O_{\leq k}|\}$, the outcome of DACB at the end of that step coincides with that of DA in the submarket composed only of individuals with serial orders in $\{|O_{\leq k-1}| + 2, ..., |O_{\leq k}|\}$ together with the individual who was rejected at the last step of Stage $k - 1$ (if $k \geq 2$) and only of objects in O_k. Since the outcome of DA does not depend on the specific serial order used, under the event that the outcome of DACB at the end of Step $|O_{\leq k}|$ coincide with that of DA in that submarket, if we permute the ordering of agents with serial orders in $\{|O_{\leq k-1}| + 2, ..., |O_{\leq k}|\}$ the outcome of DACB at Step $|O_{\leq k}|$ remains the same and so the final outcome of DACB remains the same. Thus, conditional on this event, for each agent with a serial order in $\{|O_{\leq k-1}| + 2, ..., |O_{\leq k}|\}$, the probability of not being matched by the end of Stage k is the same. Hence, since the number of such agents goes to infinity as n grows, this probability must go to 0 as n goes to infinity. Since the conditional event has a probability converging to 1 as n goes to infinity, (unconditionally) the probability of not being matched by the end of Stage k must go to 0 as n goes to infinity. ∎

In the sequel, we assume that all individuals other than i report truthfully their preferences. We partition the set of possible reports into two sets $\mathcal{T}_1$ and $\mathcal{T}_2$ as follows. $\mathcal{T}_1$ consists of the set of reports that, when restricted to objects in $O_{\geq k}$, only contain objects in O_k

within the κ first objects. $\mathcal{T}_2$ consists of the set of reports which, when restricted to objects in $O_{\geq k}$ contain some object outside O_k within the κ first objects.

We will be using the following terminology: Fix a set of possible reports $\mathcal{T}$. Given an event E_{P_i} which may depend on i's report P_i, we will say that the probability of E_{P_i} converges to 1 *uniformly across all reports in* $\mathcal{T}$ if for any $\epsilon > 0$, there is N such that for any $n \geq N$, $\Pr(E_{P_i}) \geq 1 - \epsilon$ for any report P_i in $\mathcal{T}$.

Recall that i's serial order is in $\{|O_{\leq k-1}| + 2, ..., |O_{\leq k}|\}$. In the sequel, given agent i's report, we define $p(r)$ to be the probability of obtaining the r-th ranked object within the $O_{\geq k}$ objects (we abuse notations and forget about the dependence of $p(r)$ on i's report).

Lemma S7. *If i's report is of type $\mathcal{T}_1$, then $\sum_{r=1}^{\kappa} p(r)$ converges to 1. In addition, the convergence is uniform across all possible reports in $\mathcal{T}_1$. If i's report is of type $\mathcal{T}_2$, then $\sum_{r=1}^{\ell} p(r)$ converges to 1 where $\ell \leq \kappa$ is the rank (within the $O_{\geq k}$ objects) of the first object outside O_k. In addition, the convergence is uniform across all possible reports in $\mathcal{T}_2$.*[10]

Proof. Consider $\mathcal{E}$ the event under which, independently of i's reported preferences, provided that all individuals from 1 to $|O_{\leq k-1}| + 1$ report truthfully their preferences over objects in $O_{\leq k-1}$, for each $k' = 1, ..., k-1$, the objects assigned in stage k' are exactly those in $O_{k'}$. By our argument in the proof of Proposition 1, the probability of that event tends to 1. By construction, the convergence is uniform over all of i's possible reports. From now on, let us condition on the realization of event $\mathcal{E}$. By Lemma S6, we know that with (conditional) probability going to 1, i is matched within Stage k.[11] In addition, by ex ante symmetry of objects within a given tier (given our conditioning event $\mathcal{E}$, the way i ranks objects in $O_{\leq k-1}$ does not matter), the rate at which the conditional probability goes to 1 is the same for each report in $\mathcal{T}_1$. But given that $\Pr(\mathcal{E})$ goes to 1 uniformly across all possible i's reports, the unconditional probability that i is matched within Stage k also converges to 1 uniformly across these reports. This proves the first part of the lemma.

Now, we move to the proof of the second part of the lemma. Let us consider the event that for each $k' = 1, ..., K$, all individuals other than i only rank objects in $O_{k'}$ within their κ most favorite objects in $O_{\geq k'}$. Consider as well event $\mathcal{E}$ as defined above and let $\mathcal{F}$ be

[10] More precisely, we mean that for any $\epsilon > 0$, there is N such that for any $n \geq N$, $\sum_{r=1}^{\ell} p(r) \geq 1 - \epsilon$ for any $\ell \leq \kappa$ and for any report of i which, restricted to objects in $O_{\geq k}$ contains an object outside O_k at rank ℓ (where the rank is within objects in $O_{\geq k}$).

[11] The only difference with Lemma S6 is that i's report on his preferences may not be truthful and so cannot be considered to be drawn randomly. However, it should be clear that the argument goes through as long as i's report is independent of his opponents' preferences which must be true in the environment we are considering where types are drawn independently and so where players play independently. That the conditional probability of the event "i is matched within Stage k" tends to 1 comes from the facts that the unconditional probability of the event tends to 1 and that the conditioning event $\mathcal{E}$ has a probability which tends to 1.

the intersection of these two events. By Lemma S1-(ii) as well as Proposition 1, we know that $\Pr(\mathcal{F})$ goes to 1 as n goes to infinity. By construction, the convergence is uniform over all of i's possible reports. Note that under event $\mathcal{F}$ no individual other than i will make an offer to an object outside O_k within Stage k. In addition, under $\mathcal{F}$, the probability that i is matched in Stage k goes to 1 as n goes to infinity uniformly across any report of i in $\mathcal{T}_2$.[12] Combining these observations, it must be the case that the probability that "i is matched to the object outside O_k with rank ℓ or to a better-ranked object" converges to 1 uniformly across all possible i' reports in $\mathcal{T}_2$. In addition, we know that, under $\mathcal{F}$, i can only be matched to an object in $O_{\geq k}$, hence, we get that conditional on $\mathcal{F}$, $\sum_{r=1}^{\ell} p(r)$ converges to 1 uniformly across i's possible reports. Given that $\Pr(\mathcal{F})$ goes to 1 uniformly across all possible i's reports, this statement holds for unconditional probabilities as well, as was to be shown. ∎

Since $\mathcal{T}_1$ and $\mathcal{T}_2$ cover the set of all possible reports of individual i, we get

Corollary S3. *For any i's report, we have that $\sum_{r=1}^{\kappa} p(r)$ converges to 1. Convergence is uniform across all of i's reports.*

In the sequel, we condition on event $\mathcal{E}$ defined in the proof of the above lemma, i.e., the event that, irrespective of i's reported preferences, all objects in $O_{\leq k-1}$ are gone when Stage k starts. As we already said, the probability of $\mathcal{E}$ converges to 1 uniformly across all possible i's reports. Now, we fix a type $(\hat{u}_o)_{o\in O} := (U_i(u_o, \xi_{io}))_{o\in O}$ of individual i and consider two cases depending on whether his true preference order falls into $\mathcal{T}_1$ or $\mathcal{T}_2$.

Case 1: Assume that individual i's true preference order falls into $\mathcal{T}_1$. Clearly, the expected utility of telling the truth is higher than $\sum_{r=1}^{\kappa} p_T(r)\hat{u}_r$ where $p_T(r)$ is the probability of getting the r-th best ranked object within the $O_{\geq k}$ objects (and hence within the O_k objects since i's type falls into $\mathcal{T}_1$) when reporting truthfully and $\hat{u}_r$ is i's utility for the object with rank r within the $O_{\geq k}$. By the above lemma, if i reports truthfully, then with probability going to 1 as n goes to infinity, he gets one of his κ most favorite objects within $O_{\geq k}$, thus, for some $N_1 \geq 1$, and for all $n \geq N_1$, $\sum_{r=1}^{\kappa} p_T(r) + \frac{\varepsilon}{2U(1,1)} \geq 1$.

[12] Indeed, the probability that i gets matched in Stage k under a report in $\mathcal{T}_2$ is larger than the probability that i gets matched in Stage k under the report where within the κ first objects in $O_{\geq k}$, any object outside O_k is replaced by an object in O_k. To see this, let ℓ be the rank (within $O_{\geq k}$) of the first object outside O_k under the original report. Observe that under $\mathcal{F}$, i cannot get matched to an object in a tier $k' < k$. In addition, if under the modified report, i gets matched to an object with a rank (within $O_{\geq k}$) strictly smaller than ℓ then, by definition of DACB, i will obtain the same match under the original report. Now, if i gets matched to an object with a rank (within $O_{\geq k}$) larger than ℓ then, under the original report, i applies to an object in $O_{>k}$ and so, under $\mathcal{F}$, by definition of DACB, i gets matched to that object. Since the modified report is in $\mathcal{T}_1$, and, as we already showed, the probability that i is matched in Stage k goes to 1 as n goes to infinity uniformly across any report of i in $\mathcal{T}_1$, this completes our argument.

Let us consider a lie of individual i. Given our conditioning event, what matters are the reports within objects in $O_{\geq k}$. In addition, given the symmetry of objects within each tier, it is optimal, and thus we assume, that agent i orders the objects within each tier truthfully among them.[13] Thus, we can restrict attention to a lie in which an agent lists truthfully objects in O_k for ranks 1 to $\ell - 1$ and lists an object in $O_>k$ for rank ℓ, for some ℓ (the ranks here are that within $O_{\geq k}$ objects). Clearly, if $\ell > \kappa$ then, by the previous lemma, irrespective of the exact form of the lie, for n large enough, the lie cannot benefit agent i by more than ε. Thus, we assume without loss that $\ell \leq \kappa$. By definition of DACB, for each $r = 1, ..., \ell - 1$ individual i still has probability $p_T(r)$ to get the object with rank r under the false report. But now he has probability $p_L(\ell)$ to get matched to the object in $O_{>k}$. Now, recall that for all $n \geq N_1$, $\sum_{r=1}^{\kappa} p_T(r) + \frac{\varepsilon}{2U(1,1)} \geq 1$. This implies that for all $n \geq N_1$, $\sum_{r=1}^{\kappa} p_T(r) + \frac{\varepsilon}{2U(1,1)} \geq \sum_{r=1}^{\ell-1} p_T(r) + p_L(\ell)$ and so $\sum_{r=\ell}^{\kappa} p_T(r) + \frac{\varepsilon}{2U(1,1)} \geq p_L(\ell)$. In addition, we know by the above lemma that $\sum_{r=1}^{\ell-1} p_T(r) + p_L(\ell)$ converges to 1 uniformly across all possible deviations of individual i. Since, given our conditioning event, i has zero probability to get an object in $O_{\leq k-1}$, there must exist some $N_2 \geq 1$ so that for all $n \geq N_2$, i's expected payoff when he lies is smaller than $\sum_{r=1}^{\ell-1} p_T(r)\hat{u}_r + p_L(\ell)\hat{u}_* + \frac{\varepsilon}{2}$, where $\hat{u}_*$ is the utility for agent i of the best object in $O_{>k}$. Clearly, $\hat{u}_* < \hat{u}_r$ for each $r = 1, ..., \kappa$ (recall that i's type falls into $\mathcal{T}_1$). In the sequel, we fix $n \geq \max\{N_1, N_2\}$. We obtain that, conditional on $\mathcal{E}$, the expected payoff when lying is smaller than

$$\begin{aligned}
\sum_{r=1}^{\ell-1} p_T(r)\hat{u}_r + p_L(\ell)\hat{u}_* + \frac{\varepsilon}{2} &\leq \sum_{r=1}^{\ell-1} p_T(r)\hat{u}_r + \sum_{r=\ell}^{\kappa} p_T(r)\hat{u}_* + \hat{u}_* \frac{\varepsilon}{2U(1,1)} + \frac{\varepsilon}{2} \\
&\leq \sum_{r=1}^{\ell-1} p_T(r)\hat{u}_r + \sum_{r=\ell}^{\kappa} p_T(r)\hat{u}_* + \varepsilon \\
&\leq \sum_{r=1}^{\ell-1} p_T(r)\hat{u}_r + \sum_{r=\ell}^{\kappa} p_T(r)\hat{u}_r + \varepsilon
\end{aligned}$$

where the first inequality uses the fact that $n \geq N_1$ and so that $\sum_{r=\ell}^{\kappa} p_T(r) + \frac{\varepsilon}{2U(1,1)} \geq p_L(\ell)$. The second inequality holds since $\hat{u}_* \leq U(1,1)$. The last inequality holds because $\hat{u}_* < \hat{u}_r$ for each $r = 1, ..., \kappa$. Since the expected payoff of the truth is larger than $\sum_{r=1}^{\kappa} p_T(r)\hat{u}_r$, we conclude that, conditional on $\mathcal{E}$, lying cannot make i gain more than ε whenever $n \geq \max\{N_1, N_2\}$.[14] Since $\mathcal{E}$ has a probability going to 1 uniformly across all possible deviations of individual i, a same result holds for unconditional expected payoffs.

[13] That is, for any object o and o' which both belong to the same tier k, if i prefers o to o' then i ranks o ahead of o'.

[14] Notice that by the uniform convergence result in the above lemma, N_1 and N_2 are independent on i's specific report.

Case 2: Assume that individual i has a type which falls into $\mathcal{T}_2$. Consider the κ best objects in $O_{\geq k}$ and let R be the rank (here again, the rank is taken among $O_{\geq k}$ objects) of the best object in $O_{>k}$. Clearly, the expected utility of truth-telling is higher than $\sum_{r=1}^{R} p_T(r)\hat{u}_r$ where $p_T(r)$ is the probability of getting object with rank r within the $O_{\geq k}$ objects when reporting truthfully. By the above lemma, if i reports truthfully, then with probability going to 1 as $n \to \infty$, i gets one of his R most favorite objects within $O_{\geq k}$. Thus, for some $N_1 \geq 1$, and for all $n \geq N_1$, $\sum_{r=1}^{R} p_T(r) + \frac{\varepsilon}{2U(1,1)} \geq 1$.

Let us consider a lie by individual i. Given our conditioning event, what matters are the reports within objects in $O_{\geq k}$. In addition, given the symmetry of objects within each tier, we can assume without loss of generality that agent i orders the objects within each tier truthfully among them. Let us first consider a lie where the first object in $O_{>k}$ is ranked at $R' < R$ (here again, the rankings are those within $O_{\geq k}$). Thus, one can think of such a lie as a report in which for ranks from rank 1 to rank $R'-1$ (the rank here is that within $O_{\geq k}$ objects) the agent lists truthfully among objects in O_k and for rank R', the agent lists an object in $O_>k$. In that case, by definition of DACB, for each $r = 1, ..., R'-1$ individual i still has probability $p_T(r)$ to get the object with rank r. But now he has probability $p_L(R')$ to get matched to the object in $O_{>k}$ listed at rank R'. Now, recall that for all $n \geq N_1$, $\sum_{r=1}^{R} p_T(r) + \frac{\varepsilon}{2U(1,1)} \geq 1$. This implies that for all $n \geq N_1$, $\sum_{r=1}^{R} p_T(r) + \frac{\varepsilon}{2U(1,1)} \geq \sum_{r=1}^{R'-1} p_T(r) + p_L(R')$ and so $\sum_{r=R'}^{R} p_T(r) + \frac{\varepsilon}{2U(1,1)} \geq p_L(R')$. In addition, we know by the above lemma that $\sum_{r=1}^{R'-1} p_T(r) + p_L(R')$ converges to 1 uniformly across all possible deviations of individual i. Since, given our conditioning event, i has zero probability to get an object in $O_{\leq k-1}$, there must exist some $N_2 \geq 1$ so that for all $n \geq N_2$, i's expected payoff when he lies is smaller than $\sum_{r=1}^{R'-1} p_T(r)\hat{u}_r + p_L(R')\hat{u}_* + \frac{\varepsilon}{2}$, where $\hat{u}_*$ is the utility of the object in $O_{>k}$ listed for rank R' and so must satisfy $\hat{u}_* \leq \hat{u}_r$ for each $r = 1, ..., R$. In the sequel, we fix $n \geq \max\{N_1, N_2\}$. We obtain that, conditional on $\mathcal{E}$, the expected payoff when lying is smaller than

$$\begin{aligned}
\sum_{r=1}^{R'-1} p_T(r)\hat{u}_r + p_L(R')\hat{u}_* + \frac{\varepsilon}{2} &\leq \sum_{r=1}^{R'-1} p_T(r)\hat{u}_r + \sum_{r=R'}^{R} p_T(r)\hat{u}_* + \hat{u}_* \frac{\varepsilon}{2U(1,1)} + \frac{\varepsilon}{2} \\
&\leq \sum_{r=1}^{R'-1} p_T(r)\hat{u}_r + \sum_{r=R'}^{R} p_T(r)\hat{u}_* + \varepsilon \\
&\leq \sum_{r=1}^{R'-1} p_T(r)\hat{u}_r + \sum_{r=R'}^{R} p_T(r)\hat{u}_r + \varepsilon
\end{aligned}$$

where the first inequality uses the fact that $n \geq N_1$ which implies $\sum_{r=R'}^{R} p_T(r) + \frac{\varepsilon}{2U(1,1)} \geq p_L(R')$. The second inequality holds since $\hat{u}_* \leq U(1,1)$. The last inequality holds because $\hat{u}_* \leq \hat{u}_r$ for each $r = 1, ..., R$. Since the expected payoff of the truth is larger than

$\sum_{r=1}^{R} p_T(r)\hat{u}_r$, we conclude that, conditional on $\mathcal{E}$, lying cannot benefit agent i by more than ε whenever $n \geq \max\{N_1, N_2\}$. Since, as n increases, the probability of $\mathcal{E}$ converges to 1 uniformly across all possible deviations of individual i, the same result holds for unconditional expected payoffs.

Consider next a lie which lists the best object in $O_{>k}$ for rank $R' \geq R$ (recall that the rankings are those within $O_{\geq k}$). Here again, without loss of generality, one can think of such a lie as a report in which for ranks 1 to $R-1$ (the rank here is that within $O_{\geq k}$ objects) the agent lists truthfully among objects in O_k. In that case, by definition of DACB, for each $r = 1, ..., R-1$ individual i still has probability $p_T(r)$ to get the object with rank r. But now he has probability $p_L(r)$ to get matched to the object with rank r for each $r = R, ..., R'$. Now, recall that for all $n \geq N_1$, $\sum_{r=1}^{R} p_T(r) + \frac{\varepsilon}{2U(1,1)} \geq 1$. This implies that for all $n \geq N_1$, $\sum_{r=1}^{R-1} p_T(r) + p_T(R) + \frac{\varepsilon}{2U(1,1)} \geq \sum_{r=1}^{R-1} p_T(r) + \sum_{r=R}^{R'} p_L(r)$ and so $p_T(R) + \frac{\varepsilon}{2U(1,1)} \geq \sum_{r=R}^{R'} p_L(r)$. In addition, we know by the above lemma that $\sum_{r=1}^{R-1} p_T(r) + \sum_{r=R}^{R'} p_L(r)$ converges to 1 uniformly across all possible deviations of individual i. Since, given our conditioning event, i has zero probability to get an object in $O_{\leq k-1}$, there must exist some $N_2 \geq 1$ so that for all $n \geq N_2$, i's expected payoff when he lies is less than $\sum_{r=1}^{R-1} p_T(r)\hat{u}_r + \sum_{r=R}^{R'} p_L(r)\hat{v}_r + \frac{\varepsilon}{2}$, where $\hat{v}_r \leq \hat{u}_R$ for each $r = R, ..., R'$. In the sequel, we fix $n \geq \max\{N_1, N_2\}$. Conditional on $\mathcal{E}$, the expected payoff when lying is no greater than

$$
\begin{aligned}
\sum_{r=1}^{R-1} p_T(r)\hat{u}_r + \sum_{r=R}^{R'} p_L(r)\hat{v}_r + \frac{\varepsilon}{2} &\leq \sum_{r=1}^{R-1} p_T(r)\hat{u}_r + \sum_{r=R}^{R'} p_L(r)\hat{u}_R + \frac{\varepsilon}{2} \\
&\leq \sum_{r=1}^{R-1} p_T(r)\hat{u}_r + p_T(R)\hat{u}_R + \frac{\varepsilon}{2U(1,1)}\hat{u}_R + \frac{\varepsilon}{2} \\
&\leq \sum_{r=1}^{R-1} p_T(r)\hat{u}_r + p_T(R)\hat{u}_R + \varepsilon
\end{aligned}
$$

where the first inequality uses the fact that $\hat{u}_R \geq \hat{v}_r$ for all $r = R, ..., R'$. The second inequality uses the fact that $n \geq N_1$ which implies $\sum_{r=R}^{R'} p_L(r) \leq p_T(R) + \frac{\varepsilon}{2U(1,1)}$. The last inequality follows from the fact that $\hat{u}_R \leq U(1,1)$. Since the expected payoff of the truth is larger than $\sum_{r=1}^{R} p_T(r)\hat{u}_r$, we conclude that, conditional on $\mathcal{E}$, lying cannot make i gain more than ε whenever $n \geq \max\{N_1, N_2\}$. Since, as n increases, the probability of $\mathcal{E}$ converges to 1 uniformly across all possible deviations of individual i, a same result holds for unconditional expected payoffs.

S.9 Analysis of the Extended DACB Algorithm

S.9.1 Proof of Theorem 6

The following lemma will be instrumental for the proof and is inspired from an observation by Wilson (1972) that in the environment where preferences are uncorrelated, priorities are arbitrary and the market is balanced, a modification of the DA algorithm can be studied as a standard urn model.[15] Using this analogy Wilson (1972) shows that the expected total number of offers made by individuals under DA is smaller $n \log(n)$. We strengthen this result by proving a concentration result.

Lemma S8. *Consider any arbitrary objects' priorities. In the case individuals' preferences are uncorrelated (i.e., where $U_i(o) = \xi_{i,o}$ for all i,o), for any $\gamma > 1$, with probability going to 1 as n goes to infinity, the total number of offers made by individuals under DA is smaller than $\gamma n \log(n)$.*

Proof. We start from the version of the DA mechanism as defined in McVitie and Wilson (1971). By the principle of deferred decisions, we assume that whenever an individual has an opportunity to make an offer, he makes his offer randomly to an object to which he has not yet made any offer. Now, as proposed in Wilson (1972), we modify this mechanism assuming that agents are memoryless: whenever an agents has an opportunity to make an offer he makes this offer randomly to an object in O, including those to which he has already made offers to. For each realization of individuals' preferences, the outcome is the same as with memory. Indeed, if an agent makes an offer to an object which already rejected him, he will continue to be rejected and the final outcome remains unchanged. The total number of offers when individuals have no memory must be larger than in the original case where agents have memory. We let X be the total number of offers needed for all objects in O to be matched under the mechanism where agents are memoryless. Given that $\gamma > 1$, it is enough to show that $\Pr(X \geq \gamma n \log(n)) \leq \frac{1}{n^{\gamma-1}}$. For any particular object $o \in O$, the probability that o receives no offer by time $\gamma n \log(n)$ offers are made in McVitie

[15] Wilson (1972) shows that in this environment, assuming individuals are memoryless, the number of offers needed for all objects to be matched is equivalent to the number of trials needed to collect all n coupons in the Coupon Collector Problem (in this problem, coupons are being collected, equally likely, within an urn of n different coupons).

and Wilson's algorithm[16] is

$$(1 - \frac{1}{n})^{\gamma n \log(n)} = \left((1 - \frac{1}{n})^n\right)^{\gamma \log(n)} \leq e^{-(\gamma \log(n))} = \frac{1}{n^\gamma} \tag{S0}$$

where the first term is the probability that all the first $\gamma n \log(n)$ offers have been directed to objects other than o.

Next observe that the algorithm ends (and the assignment is complete) once every object receives at least one offer. Hence, the probability that $X \geq \gamma n \log(n)$ is the probability that the algorithm (with memorlyess agents) is not complete after $n \log(n)$ offers have been made, which in turn equals the probability that at least one object has not received an offer by the time $n \log(n)$ total offers have been made. By the union bound and using Equation (S0), this latter probability is no greater $n \frac{1}{n^\gamma} = \frac{1}{n^{\gamma - 1}}$. ∎

In the sequel, for each size of the market n, we let $j(n)$ and $\kappa(n)$ be the two parameters of DACB. The following result is a straightforward implication of the above lemma.

Corollary S4. *Consider any arbitrary objects' priorities. In the case individuals' preferences are uncorrelated, if* $\liminf_{n \to \infty} \frac{j(n)\kappa(n)}{n \log(n)} > 1$ *then under DA we must have*

$$\Pr\{\textit{fewer than } j(n) \textit{ agents make more than } \kappa(n) \textit{ offers}\} \to 1$$

as $n \to \infty$.

Proof. Proceed by contradiction and assume that there is $\delta > 0$ and a sequence $n_k \to \infty$ as $k \to \infty$ such that along that sequence

$$\Pr\{\text{more than } j(n_k) \text{ individuals make more than } \kappa(n_k) \text{ offers}\} > \delta.$$

This implies that along the sequence $\{n_k\}$, there is a probability greater than $\delta > 0$ that the total number of offers made under DA is greater than $j(n_k)\kappa(n_k)$. Now, note that since $\liminf_{n \to \infty} \frac{j(n)\kappa(n)}{n \log(n)} > 1$, we must have that for some $\gamma > 1$ and for n_k large enough, $j(n_k)\kappa(n_k) > \gamma n_k \log(n_k)$. Hence, we obtain that along the sequence $\{n_k\}$, there is a probability greater than $\delta > 0$ that the total number of offers made under DA is strictly greater than $\gamma n_k \log(n_k)$ which yields a contradiction. ∎

In the sequel we fix the two parameters of the DACB mechanism to be $j(n)$ and $\kappa(n)$. Theorem 6 directly follows from the proposition below.

[16] At each stage of the algorithm, there is an individual who is rejected (the identity of the individual may depend on objects' priorities but his identity does not matter for our computations) and, using the principle of deferred decisions and the fact that individuals are memoryless, this individual makes an offer randomly to an object in O.

Proposition S2. *Fix any $k \geq 1$. As $n \to \infty$, with probability approaching one, at the end of Stage k of DACB, all objects in O_k are assigned and at most $j(n)$ objects outside O_k are assigned. In addition, for any $\epsilon > 0$*

$$\frac{|\{i \in \hat{I}_k | U_i(DACB(i)) \geq U(u_k, 1) - \epsilon\}|}{|\hat{I}_k|} \xrightarrow{p} 1$$

where $\hat{I}_k$ is the set of individuals matched at Stage k of DACB. Similarly,

$$\frac{|\{o \in \hat{O}_k | V_o(DACB(o)) \geq V(1) - \epsilon\}|}{|\hat{O}_k|} \xrightarrow{p} 1$$

where $\hat{O}_k$ is the set of objects matched at Stage k of DACB.

Proof of Proposition S2. We focus on Stage $k = 1$, as will become clear, the other cases can be treated in a similar way.

First, consider the submarket that consists of the $|O_1|$ first agents (according to the ordering given in the definition of DACB) and of all objects in O_1 objects. If we were to run standard DA just for this submarket, then because preferences are drawn *iid*, by Lemma 2 and Corollary S4, with probability approaching 1 as $n \to \infty$, at the end of (standard) DA, (a) *all* agents have made fewer than $\log^2(|O_1|) \leq \log^2(n)$ offers and (b) fewer than $j(|O_1|) \leq j(n)$ individuals have made more than $\kappa(|O_1|) \leq \kappa(n)$ offers.

Consider now the original market. For any $\delta > 0$, since $k(n) = o(n)$, we must have that $k(n) \leq \delta |O_1|$ for any n large enough. Hence, by Lemma S1-(ii) the event that for each agent's $\max\{\kappa(n), \log^2(n)\}$ favorite objects are in O_1 has probability approaching 1 as $n \to \infty$. Let us condition on this event, labeled $\mathcal{E}$.

We now show that, conditional on $\mathcal{E}$, with probability approaching 1 as $n \to \infty$, *all* objects in O_1 and no more than $j(n)$ objects outside O_1 are assigned by the end of Stage 1. Note that under our conditioning event $\mathcal{E}$, the distribution of individuals' preferences over objects in O_1 is the same as the unconditional one (and the same is true for the distribution of objects' priorities over individuals). Given event $\mathcal{E}$, as long as each agent has made fewer than $\log^2(n)$ offers (which ensures that offers are only made to O_1 objects) and fewer than $j(n)$ individuals have made more than $\kappa(n)$ offers (which ensures that the end of Stage 1 is not triggered), the $|O_1|$ first steps of DACB proceed exactly in the same way as DA in the submarket composed of the $|O_1|$ first agents (according to the ordering used in DACB) and of all objects in O_1. Hence, as mentioned above, with probability going to 1 as $n \to \infty$, we then reach the end of Step $|O_1|$ of DACB before Stage 1 ends (i.e., before more than $j(n)$ individuals applied to their $\kappa(n)$ most favorite object). Hence, conditional on $\mathcal{E}$, with probability going to 1, all objects in O_1 are assigned before Stage 1 ends. In addition, under event $\mathcal{E}$, if more than $j(n)$ individuals make offers to objects outside O_1 before Stage 1 ends, then more than $j(n)$ individuals make more than $\kappa(n) = o(n)$ offers which is not possible by definition of a stage in DACB. Hence, under event $\mathcal{E}$, Stage 1 must end before more than $j(n)$ objects outside O_1 are assigned. Now, since $\Pr(\mathcal{E}) \to 1$ as $n \to \infty$, we obtain that at the end of stage 1 of DACB, all objects in O_1 are assigned and at most $j(n)$ objects outside O_1 are assigned with probability going to 1. This completes the proof of the first part of Proposition S2.

Now, we move to the proof of the second part of Proposition S2. We fix any $\epsilon > 0$ and $\gamma < 1$ and want to show that as $n \to \infty$,

$$\Pr\left\{\frac{|\{i \in \hat{I}_1 | U_i(DACB(i)) \geq U(u_1, 1) - \epsilon\}|}{|\hat{I}_1|} > \gamma\right\} \longrightarrow 1$$

and

$$\Pr\left\{\frac{|\{o \in \hat{O}_1 | V_o(DACB(o)) \geq V(1) - \epsilon\}|}{|\hat{O}_1|} > \gamma\right\} \longrightarrow 1.$$

In the sequel, we condition on event $\mathcal{E}$. Recall that, with probability going to 1, the number of individuals matched in Stage 1 is between $|O_1|+1$ and $|O_1|+j(n)$ and all these individuals except possibly for $j(n)$ of them obtain an object within their $\kappa(n)$ most favorite objects. By Lemma S1-(i) this implies that, with probability going to 1, all these individuals but potentially $j(n) = o(n)$ of them enjoy a payoff above $U(u_1, 1) - \epsilon$. As we have shown, with probability going to 1, the first $|O_1|$ Steps of Stage 1 of DACB proceed exactly in the same way as DA in the submarket that consists of the $|O_1|$ first agents (according to the ordering used in DACB) and of all objects in O_1. We first note that, by Lemma 2, under DA in this submarket, with probability going to 1 the proportion of objects in O_1 with a rank smaller than $\frac{2}{1-\gamma}|O_1|/\log(|O_1|)$ is larger than γ. To see this, suppose to the contrary that with probability bounded away from 0, as the market grows, the proportion of objects with a rank above $\frac{2}{1-\gamma}|O_1|/\log(|O_1|)$ is more than $1-\gamma$. Then, with probability bounded away from 0, as the market grows: $\frac{1}{|O_1|}\sum_{o \in O_1} R_o^{DA} > \frac{1}{|O_1|}(1-\gamma)|O_1|\frac{2}{1-\gamma}(|O_1|/\log(|O_1|)) = 2|O_1|/\log(|O_1|)$ which yields a contradiction to Lemma 2. Hence, we obtain that with probability going to 1 by the end of Step $|O_1|$ of Stage 1 of DACB, the proportion of objects in O_1 with a rank smaller than $\frac{2}{1-\gamma}|O_1|/\log(|O_1|)$ is larger than γ. Given that for any $\delta > 0$, for n large enough, $|O_1|/\log(|O_1|) \leq \delta\,|I|$, by Lemma S1-(iii), we must also have that, with probability going to 1, the proportion of objects o in O_1 with $V(DACB(o)) \geq 1-\epsilon$ is above γ. Since objects in O_1 will have received even more offers at the end of Stage 1, it must still be that, with probability going to 1, the proportion of objects in O_1 for which $V(DACB(o)) \geq 1-\epsilon$ is above γ when n is large enough. We ignore the remaining objects matched in Stage 1 since there are fewer than $j(n) = o(n)$ such objects. Thus, for $k = 1$, the second statement in Proposition S2 is proved provided that our conditioning event $\mathcal{E}$ holds. Since, this event has probability going to 1 as $n \to \infty$, the result must hold even without the conditioning. Thus, we have proved Proposition S2 for the case $k = 1$.

Consider next Stage $k > 1$. The objects remaining in Stage k have received no offers in Stages $1, ..., k-1$ (or else the objects would have been assigned in those stages). Hence, by the principle of deferred decisions, we can assume that the individuals' preferences over those objects are yet to be drawn in the beginning of Stage k. Similarly, we can assume that priorities of those objects are also yet to be drawn. Put in another way, conditional on Stage $k-1$ being over, we can assume without loss that the distribution of preferences and priorities is the same as the unconditional one. Thus, we can consider the market composed of the individuals and objects not matched in previous stages. We can set O_1 to be equal to the set of remaining objects in O_k, O_2 to be equal to the set of remaining objects in O_{k+1}, etc... (with high probability, the cardinality of each tier defined in this way is linear in n, i.e., between $|O_\ell| - (\ell - 1)j(n)$ and $|O_\ell|$ for each ℓ) so the exact same reasoning as above completes the argument. ∎

S.9.2 Incentives under the Extended Algorithm

In the sequel, we consider the extended version of DACB with parameters $j(n)$ and $\kappa(n)$. We first slightly strengthen our assumption that the serial orders admit some *basic uncertainty* from the agents' perspective: for each sequence of sets $E^n \subset \{1, \ldots, n\}$ such that $\lim |E^n|/n$ goes to 0 as n goes to infinity, we assume that the probability that any agent i receives a serial order in E^n goes to zero as $n \to \infty$.

We show that the following result.

Theorem S4. *Consider a sequence of economies satisfying* $\liminf_{n\to\infty} \frac{j(n)\kappa(n)}{n \log(n)} > 1$ *and* $j(n)$ *and* $\kappa(n)$ *are* $o(n)$. *Fix any* $\epsilon > 0$. *Under DACB, there exists* $N > 0$ *such that for all* $n > N$, *truthtelling is an interim* ϵ*-Bayes-Nash equilibrium.*

The proof is rather similar to that of Theorem 5. The two main lemmas (Lemmas S7 and S6) from the proof of Theorem 5 have to be adapted. The rest of the proof is in essence the same and is thus omitted.

Let us fix $\varepsilon > 0$ and $k = 1, ..., K$. Assume that the ordering of DACB gives to agent i a serial order in $\{|O_{\leq k-1}| + 1 + j(n), ..., |O_{\leq k}|\}$ with the convention that $|O_{\leq 0}| + 1 + j(n) = 1$. We show that there is $N \geq 1$ such that for any $n \geq N$, for any vector of cardinal utilities $(\hat{u}_o)_{o \in O} := (U_i(u_o, \xi_{io}))_{o \in O}$, i cannot gain more than ε by deviating given that everyone else reports truthfully. As will be clear, the argument does not depend on the specific serial order of i within $\{|O_{\leq k-1}| + 1 + j(n), ..., |O_{\leq k}|\}$ and so given that there are finitely many tiers, N can be taken to be uniform across all individuals with serial order in $\cup_{k=1}^{K}\{|O_{\leq k-1}|+1+j(n),...,|O_{\leq k}|\}$. Hence, conditional on the event that i's serial order is in $\{|O_{\leq k-1}|+1+j(n),...,|O_{\leq k}|\}$ for some $k=1,...,K$, it will follow that for any $n \geq N$, for any vector of cardinal utilities, i cannot gain more than ε by deviating given that everyone else reports truthfully. Now, given our assumption on the distribution from which the ordering of DACB is drawn, the probability of the conditioning event goes to 1. Hence, for any $n \geq N$, for any vector of cardinal utilities $(\hat{u}_o)_{o\in O} := (U_i(u_o, \xi_{io}))_{o\in O}$, i cannot gain more than ε by deviating given that everyone else reports truthfully – which shows the desired result.

Lemma S9. *Let us assume that* $\liminf_{n\to\infty} \frac{j(n)\kappa(n)}{n\log(n)} > 1$. *Consider the DACB mechanism. Fix any* $k=1,\ldots,K$ *and any agent* i *with a serial order in* $\{|O_{\leq k-1}|+1+j(n),...,|O_{\leq k}|\}$. *Assuming all agents report truthfully, the probability that* i *is matched at Stage* k *to one of his* κ *most favorite choices within remaining objects at that stage converges to 1 as* $n\to\infty$.

Proof. By the argument in the proof of Proposition S2, we know that with probability approaching 1 as n goes to infinity, Step $|O_{\leq k}|$ ends under DACB[17] and, for any agent i with a serial order in $\{|O_{\leq k-1}|+1+j(n),...,|O_{\leq k}|\}$, the outcome of DACB at the end of that step coincides with that of DA in the submarket composed only of remaining individuals at that stage with a serial order below $|O_{\leq k}|$ (which contains all individuals with serial order in $\{|O_{\leq k-1}|+1+j(n),...,|O_{\leq k}|\}$) and the remaining objects in O_k. Since the outcome of DA does not depend on the specific linear order used, under the event that the outcome of DACB at the end of Step $|O_{\leq k}|$ coincide with that of DA in that submarket, if we switch the ordering (of the linear order of DACB) of agents with serial order in $\{|O_{\leq k-1}|+j(n)+1,...,|O_{\leq k}|\}$ the outcome of DACB at Step $|O_{\leq k}|$ remains the same and so the final outcome of DACB remains the same. Thus, conditional on this event, for each agent with a serial order in $\{|O_{\leq k-1}|+j(n)+1,...,|O_{\leq k}|\}$, the probability of either not being matched by the end of Stage k or of being one of the $j(n)$ individuals making more than κ offers is the same. Hence, since the number of such agents goes to infinity as n grows, this probability must go to 0 as n goes to infinity. Since the conditional event has a probability converging to 1 as n goes to infinity, (unconditionally) this probability must go to 0 as n goes to infinity. ∎

In the sequel, we let $\hat{O}_{\geq k}$ be the set of available objects in Stage k. Note that this does not depend on i's reports (given that i's serial order is in $\{|O_{\leq k-1}|+1+j(n),...,|O_{\leq k}|\}$). We assume that all individuals other than i report truthfully their preferences. We partition the set of i's possible reports into two sets $\mathcal{T}_1$ and $\mathcal{T}_2$ as follows. $\mathcal{T}_1$ consists of the set of

[17] Here, when counting the number of steps which occurred by the end of Stage k, we consider the total number of steps from the beginning of Stage 1. Hence, we say that Step $|O_{\leq k}|$ of Stage k ends if, from the beginning of Stage 1, $|O_{\leq k}|$ steps have occurred.

i's reports that, when restricted to objects in $\hat{O}_{\geq k}$, only contain objects in O_k within the κ first objects. $\mathcal{T}_2$ consists of the set of i's reports which, when restricted to objects in $\hat{O}_{\geq k}$, contain some object outside O_k within the κ first objects.

We will again be using the following terminology: Fix a set of possible reports $\mathcal{T}$. Given an event E_{P_i} which may depend on i's report P_i, we will say that the probability of E_{P_i} converges to 1 *uniformly across all reports in* $\mathcal{T}$ if for any $\epsilon > 0$, there is N such that for any $n \geq N$, $\Pr(E_{P_i}) \geq 1 - \epsilon$ for any report P_i in $\mathcal{T}$.

Recall that i's serial order is in $\{|O_{\leq k-1}| + 1 + j(n), ..., |O_{\leq k}|\}$. In the sequel, given agent i's report, we define $p(r)$ to be the probability of obtaining the r-th ranked object within the $\hat{O}_{\geq k}$ objects (we abuse notations and forget about the dependence of $p(r)$ on i's report).

Lemma S10. *If i's report is of type $\mathcal{T}_1$, then $\sum_{k=1}^{\kappa} p(k)$ converges to 1. In addition, the convergence is uniform across all possible reports in $\mathcal{T}_1$. If i's report is of type $\mathcal{T}_2$, then $\sum_{k=1}^{\ell} p(k)$ converges to 1 where $\ell \leq \kappa$ is the rank (within the $\hat{O}_{\geq k}$ objects) of the first object outside O_k. In addition, the convergence is uniform across all possible reports in $\mathcal{T}_2$.*[18]

Proof. Consider $\mathcal{E}$ the event under which, independently of i's reported preferences, provided that all individuals from 1 to $|O_{\leq k-1}| + j(n)$ report truthfully their preferences, for each $k' = 1, ..., k-1$, the objects assigned in Stage k' contain all those in $O_{k'}$ and contain no more than $j(n)$ objects from $O_{>k'}$. By our argument in the proof of Proposition 1, the probability of this event tends to 1. By construction, the convergence is uniform over all of i's possible reports. From now on, let us condition on the realization of event $\mathcal{E}$. By Lemma S9, we know that with (conditional) probability going to 1 as $n \to \infty$, i makes fewer than κ offers and is matched within Stage k. In addition, by ex ante symmetry of objects within a given tier (given our conditioning event $\mathcal{E}$, the way i ranks objects in $O_{\leq k-1}$ does not matter), the rate at which the conditional probability goes to 1 is the same for each report in $\mathcal{T}_1$. But given that $\Pr(\mathcal{E})$ goes to 1 uniformly across all possible i's reports, the unconditional probability that i is matched within Stage k to an object within his κ most favorite in $\hat{O}_{\geq k}$ also converges to 1 uniformly across these reports. This proves the first part of the lemma.

We next move to the proof of the second part of the lemma. Let us consider the event that for each $k' = 1, ..., K$, all individuals other than i only rank objects in $O_{k'}$ within their κ most favorite objects in $\hat{O}_{\geq k'}$ (recall that with probability approaching 1 as n goes to infinity, the size of $\hat{O}_{\geq k'} \cap O_{k'}$ is linear in n). Consider as well event $\mathcal{E}$ as defined above

[18] More precisely, we mean that for any $\epsilon > 0$, there is N such that for any $n \geq N$, $\sum_{k=1}^{\ell} p(k) \geq 1 - \epsilon$ for any $\ell \leq \kappa$ and for any report of i which, restricted to objects in $O_{\geq k}$ contains an object outside O_k at rank ℓ (where the rank is within objects in $O_{\geq k}$).

and let $\mathcal{F}$ be the intersection of these two events. By Lemma S1-(ii) as well as Proposition 1, we know that $\Pr(\mathcal{F})$ goes to 1 as n goes to infinity. By construction, the convergence is uniform over all of i's possible reports. Note that under event $\mathcal{F}$ no individual other than i will make an offer to an object outside O_k within Stage k. In addition, under $\mathcal{F}$, the probability that i is matched in Stage k and obtains an object with his κ most favorite with remaining objects goes to 1 as n goes to infinity uniformly across any report of i in $\mathcal{T}_2$.[19] Combining these observations, it must be the case that the probability that "i is matched to the object outside O_k with rank ℓ or to a better object" converges to 1 uniformly across all possible i' reports in $\mathcal{T}_2$. In addition, we know that under $\mathcal{F}$, agent i can only be matched to an object in $\hat{O}_{\geq k}$, hence, $\sum_{k=1}^{\ell} p(k)$ converges to 1 uniformly across i's possible reports. Given that $\Pr(\mathcal{F})$ goes to 1 uniformly across all possible i's reports, this statement holds for unconditional probabilities as well, as was to be shown. ∎

S.10 Proof of Theorem 7

Recall that in the two tier environment under consideration (i.e., $K = 2$), given $x \in (0, 1)$, we denote σ_x for the symmetric profile of strategies where each agent of each type lists first his $x\kappa(n)$ most favorite objects in O_1 and then his $(1 - x)\kappa(n)$ most favorite objects in O_2.

For each integer n, define $x(n) := 1 - \sqrt{\log^2(x_2 n)/\kappa(n)}$. Observe that, since we assumed that $\kappa(n)/\log^2(n) \to \infty$ as $n \to \infty$, $x(n)$ converges to 1 as n grows large. We show that the sequence of symmetric strategies $\{\sigma_{x(n)}\}_n$ satisfies the following two properties. (1) For any $\varepsilon > 0$, and any n large enough, $\sigma_{x(n)}$ is an ex-ante ε-Bayes Nash equilibrium; (2) the induced matching outcome is asymptotically efficient and asymptotically stable.

For the proof, we first establish the following claim:

Claim: *Given the sequence of profiles of strategies $\{\sigma_{x(n)}\}_n$, the probability that all agents*

[19] Indeed, the probability that i gets matched in Stage k to one of his κ most favorite object in $\hat{O}_{\geq k}$ under a report in $\mathcal{T}_2$ is larger than the probability that i gets matched in Stage k to one of his κ most favorite object in $\hat{O}_{\geq k}$ under the report where within the κ first objects in $\hat{O}_{\geq k}$, any object outside O_k is replaced by an object in O_k. To see this, let ℓ be the rank (within $\hat{O}_{\geq k}$) of the first object outside O_k under the original report. Observe that under $\mathcal{F}$, i cannot get matched to an object in a tier $k' < k$. In addition, if under the modified report, i gets matched to an object with a rank (within $\hat{O}_{\geq k}$) strictly smaller than ℓ then, by definition of DACB, i will obtain the same match under the original report. Now, if i gets matched to an object with a rank (within $\hat{O}_{\geq k}$) in $\{\ell, \ldots, \kappa\}$ then, under the original report, i applies to an object in $O_{>k}$ and so, under $\mathcal{F}$, by definition of DACB, i gets matched to that object. Since the modified report is in $\mathcal{T}_1$, and, as we already showed, the probability that i is matched in Stage k to one of his κ most favorite object in $\hat{O}_{\geq k}$ goes to 1 as n goes to infinity uniformly across any report of i in $\mathcal{T}_1$, this completes our argument.

are matched goes to 1 *as* n *goes to infinity.*

Proof. Given an n-economy, recall that under the profile of strategies $\sigma_{x(n)}$, the assignment of DA (with truncation $\kappa(n)$) can be obtained in two steps. In the first step, we run DA with only first tier objects and the agents ranking only $x(n)\kappa(n)$ objects. To complete the matching, in the second step, we run DA with only second tier objects and all unmatched individuals ranking only $(1 - x(n))\kappa(n)$ objects.[20] Now, in Step 1, there are more individuals than objects and ordinal preferences are drawn iid uniformly. We will show that with probability going to 1 in this unbalanced market all objects are matched. Pick randomly $x_1 n$ individuals. We obtain a new market composed of these $x_1 n$ individuals together with the $x_1 n$ objects in O_1. In this balanced market, the number of offers received by each object is weakly smaller than in the market with all n individuals and the $x_1 n$ objects in O_1. Thus, it is enough for our purpose to show that, with probability going to 1, in this balanced market all objects are matched. In order to show this, consider DA without any constraint on offers. We know, by Pittel (1992), that in this balanced market, with probability going to 1 as n goes to infinity, all objects in O_1 are matched before agents make more than $\log^2(x_1 n)$ offers. In Step 1, our mechanism is DAC with the length of ROL at $x(n)\kappa(n)$. Because $\kappa(n)/\log^2(n) \to \infty$ as $n \to \infty$ and $x(n)$ goes to 1 as $n \to \infty$, there is $N > 0$ such that $x(n)\kappa(n) \geq \log^2(x_1 n)$ for any $n > N$ (recall that x does not depend on n). Combining the two previous observations, it follows that, in the balanced market, with probability going to 1 as n goes to infinity, all objects in O_1 are matched, as claimed.

Now, in Step 2 we have all remaining individuals and all objects in O_2. Since we only have objects in O_1 in Step 1, the number of individuals who remain unmatched at the end of Step 1 must be (weakly) greater than $|O| - |O_1| = |O_2|$. Hence, Step 2 is a (weakly) unbalanced market where there are more individuals than objects and where the length of each agent's ROL is $(1 - x(n))\kappa(n)$. In addition, by the principle of deferred decisions, we can simply assume that preferences/priorities are yet to be drawn. Finally, we observe that $(1 - x(n))\kappa(n)/\log^2(x_2 n) = \sqrt{\kappa(n)/\log^2(x_2 n)}$ goes to ∞ as n grows large. So for n large enough, $(1 - x(n))\kappa(n)$ is greater than $\log^2(x_2 n)$. Thus, we can mimic the reasoning we just made for Step 1 to show that with probability going to 1 as n grows, all objects in O_2 are matched.

By combining the two above results (for Step 1 and 2), we conclude that the probability that all objects (and so that all agents) are matched goes to 1 as n goes to infinity, as claimed. ∎

Because agents are symmetric, by the previous result, for each individual, the probability

[20] There may exist O_1 objects which are not assigned by the end of the first step. We do not have to consider them in the second step because they would in any case receive no offer in that stage (since all agents remaining in the second step have exhausted all their offers to objects in O_1).

of being matched an O_1 object and that of being matched an O_2 object converges to x_1 and x_2 respectively. We state this in the following corollary.

Corollary S5. *Given the sequence of profiles of strategies $\{\sigma_{x(n)}\}_n$, for each $k = 1, 2$, the probability that an agent is matched to an object in O_k goes to x_k as n goes to infinity.*

The above corollary can be used to prove that the strategy profiles in the sequence have a desirable incentive property.

Proposition S3. *For any $\varepsilon > 0$, for any n large enough, $\sigma_{x(n)}$ is an ex-ante ε-Bayes Nash equilibrium in the n-economy.*

Proof. Fix any $\varepsilon > 0$. Without loss of generality, assume that $\varepsilon > 0$ is small enough so that $U(u_1, 1) - \varepsilon > U(u_2, 1)$. We claim that for n large enough $\sigma_{x(n)}$ is an ex-ante ε-Bayes Nash equilibrium. Let n be large enough so that $(1 - x(n))U(u_1, 1) \leq \varepsilon$.

Let E_i be the event that agent i realizes utility of at least $U(u_k, 1) - \varepsilon$ from each of her $\kappa(n)$ most favorite objects in O_k for $k = 1, 2$. This last condition ensures that under event E_i, individual i prefers each object in O_1 that he ranks to any object in O_2 (since $U(u_1, 1) - \varepsilon > U(u_2, 1)$). Because $\kappa(n) = o(n)$, by Lemma S1, the probability of E_i goes to 1 as n increases. Thus, the expected payoffs conditional on E_i of agent i converge to the (unconditional) ex-ante payoffs.

Note that, by symmetry, the probability of getting matched an O_1 object for individual i does not depend on ξ.[21] Hence, it must be the same as the ex-ante probability of being matched to an O_1 object which, by Corollary S5, converges to x_1 under the profile of strategies $\sigma_{x(n)}$. Similarly, the probability that ξ gets matched to an O_2 object converges to x_2 and the probability that he gets unmatched converges to 0. Fix n large enough so that the probability of being unmatched given an arbitrary type ξ is bounded from above by $\varepsilon / U(u_2, 1)$.

In the sequel, we fix an agent and consider a deviation from the strategy prescribed by $\sigma_{x(n)}$. We fix a realization of type $\xi \in E_i$ and assume without loss of generality that, under the deviating strategy, ξ lists first his $z(n)$ most favorite objects in O_1 and then his $\kappa(n) - z(n)$ most favorite objects in O_2.[22] There are two cases to consider.

Case 1. $z(n) \leq x(n)\kappa(n)$. One of the following events must be true.

1a. Under the profile of strategies $\sigma_{x(n)}$, ξ gets matched to an O_1 object within his $z(n)$ most favorite objects. Here, the deviation has no impact on ξ's assignment.

[21] Agents with types ξ and ξ' rank the same number of objects in O_1 and the same number of objects in O_2. In addition, objects within each tier are ex-ante symmetric.

[22] This follows from the ex ante symmetry of the objects within each tier and the use of symmetric strategies adopted by the opponents. Listing objects within a tier untruthfully or dropping a more preferred object within a tier can only do worse.

1b. Under the profile of strategies $\sigma_{x(n)}$, ξ gets matched to an O_1 object between his $z(n)+1$ and $x\kappa(n)$ most favorite objects. Here, the deviation entails a payoff loss strictly greater than $U(u_1, 1) - \varepsilon - U(u_2, 1) > 0$ (recall that each object in O_1 listed under the profile of strategies σ_x all yields payoff of least $U(u_1, 1) - \varepsilon > U(u_2, 1)$).

1c. Under the profile of strategies σ_x, ξ gets matched to an O_2 object. Under the deviating strategy, ξ cannot be assigned an object in O_1. While the exact impact of the deviating strategy on the size of the set of participants in Step 2 is not easy to analyze, the gain (if any) from the deviating strategy is bounded by $U(u_2, 1) - (U(u_2, 1) - \varepsilon) = \varepsilon$ since $\xi \in E_i$.

1d. Under the profile of strategies $\sigma_{x(n)}$, ξ gets unmatched. Under the deviating strategy, ξ cannot be assigned an object in O_1. The gain of the deviating strategy is at most $U(u_2, 1)$.

Hence, the expected gain of the deviating strategy is bounded from above by $\Pr\{1c\}\varepsilon + \Pr\{1d\}U(u_2, 1)$. By construction, $\Pr\{1d\}U(u_2, 1) \leq \varepsilon$ and so the expected gain of the deviating strategy is at most 2ε.

Case 2. $z(n) \geq x(n)\kappa(n)$. One of the following events must occur.

2a. Under the deviating strategy, ξ gets matched to an O_1 object within his $x(n)\kappa(n)$ most favorite objects. Here, the deviation has no impact on ξ's assignment.

2b. Under the deviating strategy, ξ gets matched to an O_1 object between his $x(n)\kappa(n)+1$ and $z(n)$ most favorite objects. Here, the deviation yields a payoff gain of at most $U(u_1, 1) > 0$ (the worst case scenario being that ξ gets unassigned under $\sigma_{x(n)}$).

2c. Under the deviating strategy, ξ gets matched to an O_2 object while under the strategy associated with $\sigma_{x(n)}$, ξ gets assigned an object in O_2 (he cannot be assigned an object in O_1). The gain (if any) obtained using the deviating strategy is at most $U(u_2, 1) - (U(u_2, 1) - \varepsilon) = \varepsilon$ since $\xi \in E_i$.

2d. Under the deviating strategy, ξ gets matched to an O_2 object while under $\sigma_{x(n)}$, ξ gets unassigned. The gain obtained using the deviating strategy is at most $U(u_2, 1)$.

2e. Under the deviating strategy, ξ gets unmatched. Trivially, there can be no gain from using the deviation.

Hence, the expected gain of the deviating strategy is bounded from above by $\Pr\{2b\}U(u_1, 1) + \Pr\{2c\}\varepsilon + \Pr\{2d\}U(u_2, 1)$. By construction, $\Pr\{2d\}U(u_2, 1) \leq \varepsilon$. In addition, we show below that $\Pr\{2b\}U(u_1, 1) \leq \varepsilon$ which implies that the expected gain of the deviating strategy is bounded from above by 3ε.

Thus, it remains to show that $\Pr\{2b\}U(u_1, 1) \leq \varepsilon$. Recall that $\{2b\}$ is defined as the event where, under the deviating strategy, ξ gets matched to an O_1 object between his $x(n)\kappa(n)+1$ and $z(n)$ most favorite objects. Recall the basic property that if o and o' are in O_1 and o is listed ahead of o' in i's list then the probability of getting assigned object

o is higher than that of being assigned o'. Hence, the probability that under the deviating strategy, ξ is assigned an object between his $x(n)\kappa(n)+1$ and $z(n)$ most favorite objects is bounded from above by $1-x(n)$.[23] Thus, $\Pr\{2b\} \leq 1-x(n)$. Our assumption that $(1-x(n))U(u_1,1) \leq \varepsilon$ completes the argument.

To recap, there is n large enough so that, given any realization $\xi \in E_i$, the expected gain from deviating is at most 3ε. Since this is true for any $\xi \in E_i$, the expected gains from deviating conditional on E_i must be bounded by 3ε. Since payoffs are bounded and $\Pr(E_i)$ goes to 1 as n grows, we are done. ∎

Proof of Theorem 7. Fix any $\varepsilon > 0$ and let n be large enough so that $\sigma_{x(n)}$ is an ex ante ε-equilibrium (this is well-defined by Proposition S3). We claim that the resulting outcome is asymptotically efficient and asymptotically stable.

Asymptotic efficiency. We know by Claim 1 that the probability that all agents are matched goes to 1 as n goes to infinity. In addition, for any $\delta > 0$, as we showed in Lemma S1, the probability that all agents list only objects with which they enjoy idiosyncratic payoffs greater than $1-\delta$ goes to 1 as grows. Taken together, these two statements imply that, for any $\delta > 0$, with probability going to 1 as n grows, all agents get matched to object with which they enjoy idiosyncratic payoffs greater than $1-\delta$. Hence, the outcome is asymptotically efficient.

Asymptotic stability. To show the asymptotic stability of the induced outcome, we come back to the proof of the last claim. Recall that the outcome of DA (with truncation $\kappa(n)$) was obtained in two steps. In Step 1, the submarket is composed of all n individuals and only objects in the first tier O_1. We are in an environment where there are more individuals than objects. Let us pick $x_1 n$ individuals randomly and consider the balanced market composed of these $x_1 n$ individuals and the $x_1 n$ objects in the first tier O_1. Clearly, under our mechanism, each object is weakly worse-off in this new market. If we run standard DA in this market, we know from Pittel (1992) that with probability going to 1, the algorithm will end before agents make more than $\log^2(x_1 n)$ offers. In Step 1, our mechanism is DA with truncation $x(n)\kappa(n)$. Because $\kappa(n)/\log^2(n) \to \infty$ as $n \to \infty$ and $x(n)$ goes

[23] To see this, let us denote by p_ℓ the probability of being matched an object listed in the ℓth position for $\ell = 1, ..., z(n)$. We claim that $\sum_{\ell=x(n)\kappa(n)+1}^{z(n)} p_\ell \leq 1-x(n)$. Suppose to the contrary that $\sum_{\ell=x(n)\kappa(n)+1}^{z(n)} p_\ell > 1-x(n)$. Then because $\{p_\ell\}$ is a decreasing sequence, we would have $p_{x(n)\kappa(n)+1} > \frac{1-x(n)}{z(n)-x\kappa(n)}$. Again because $\{p_\ell\}$ is a decreasing sequence, we obtain that $\sum_{\ell=1}^{x(n)\kappa(n)} p_\ell > x(n)\frac{1-x(n)}{z(n)/\kappa(n)-x(n)}$ where the term $\frac{1-x(n)}{z(n)/\kappa(n)-x(n)}$ is greater than 1 because $z(n)/\kappa(n) \leq 1$. Thus, $\sum_{\ell=1}^{x(n)\kappa(n)} p_\ell > x(n)$ and so eventually we get $\sum_{\ell=1}^{z(n)} p_\ell > x(n)+1-x(n) = 1$, a contradiction.

to 1 as n grows large, there exists $N > 0$ such that $x(n)\kappa(n) \geq \log^2(n)$ for any $n > N$. Combining the two previous observations, in the balanced submarket, with probability going to 1 as n goes to infinity, the matching given by DA and the matching given by DA with truncation $x(n)\kappa(n)$ are the same. By Lemma 3, in the balanced submarket, for any $\delta > 0$, the proportion of objects in O_1 receiving a (idiosyncratic) payoff greater than $1 - \delta$ converges in probability to 1. Thus, in the whole market where there are (weakly) more individuals than objects this must remain true. Now, to treat objects in O_2, we must consider the second step. Again, there are (weakly) more individuals than objects and where preferences/priorities over O_2 objects (i.e. remaining ones) are yet to be drawn. In addition, as we already claimed, $(1 - x(n))\kappa(n)/\log^2(x_2 n) = \sqrt{\kappa(n)/\log^2(x_2 n)}$ goes to ∞ as n grows large, and so, for n large enough, $(1 - x(n))\kappa(n)$ is greater than $\log^2(x_2 n)$. Hence, the same argument as for O_1 objects can be applied to O_2 objects To recap, for any $\delta > 0$, the proportion of objects in O receiving a (idiosyncratic) payoff greater than $1 - \delta$ converges in probability to 1. This completes the proof of the asymptotic stability of the induced outcome. ∎

S.11 Simulations Based on Randomly Generated Data

This section is organized in three parts. We start by providing simulations showing that our limit results hold more generally even for moderate size markets. We next show that the the results are robust to more general forms of preferences and priorities. Importantly, we explain how DACB has to be modified in this richer environment by defining properly the serial orders given to agents.

Finally, for fixed market sizes, we study how DACB(κ, j) performs (for different values (κ, j)) and show how its performance evolves when we vary correlation in agents' preferences. We see the large compromise between efficiency and stability that can be attained by DACB when (κ, j) vary. By doing so, we will clearly see how a small departure from exact stability (resp., efficiency) can allow for a significant increase in efficiency (resp., stability) performance. Finally, we will see how this trade-off evolves when correlation increases. Overall, we observe that the *relative performance* of DACB increases when correlation increases.

S.11.1 Robustness to market sizes

Figure 1 shows the utilitarian welfare—more precisely the average idiosyncratic utility enjoyed by the agents—achieved by alternative algorithms, including DACB with $\kappa(n) = \log^2(n)$, under the assumption that $U(u_o, \xi_{i,o}) = u_o + \xi_{i,o}$ and $V(\eta_{i,o}) = \eta_{i,o}$, and that each of $u_o, \xi_{i,o}$ and $\eta_{i,o}$ are distributed uniformly from $[0, 1]$. Importantly, the welfare is measured under the varying market size ranging from $n = 10$ to $10,000$.[24] As expected, the TTC

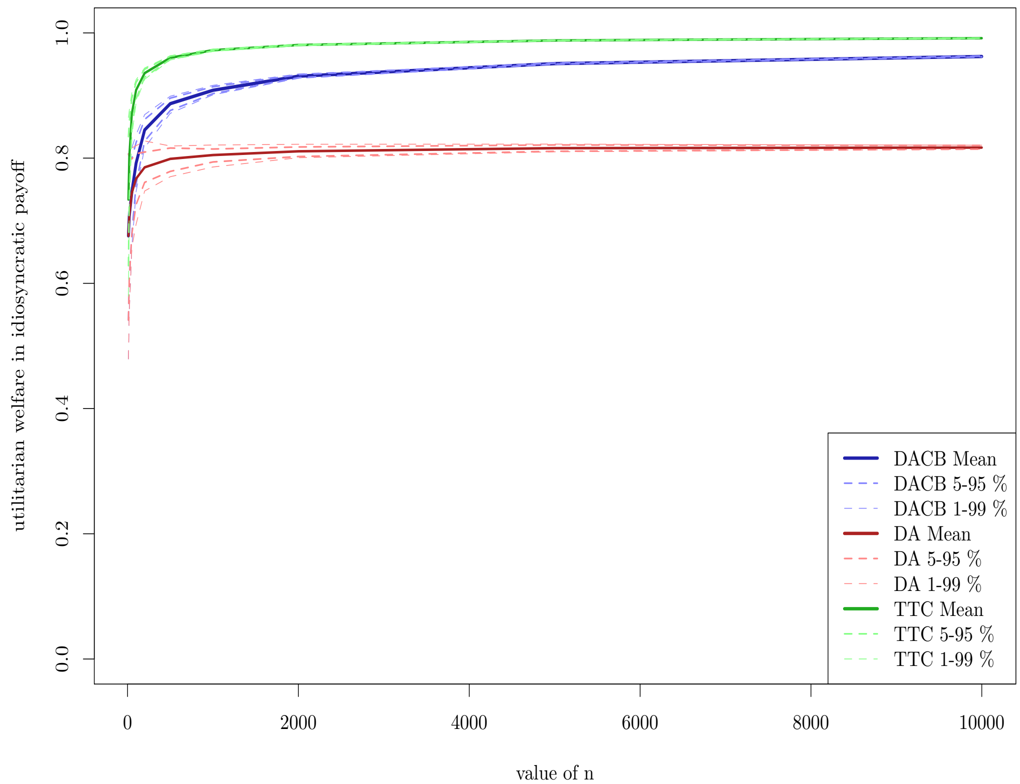


Figure 1: Utilitarian welfare under alternative mechanisms

achieves higher utilitarian welfare, followed by DACB, and DA, and they all increase with the market size. But the levels of the utilitarian welfare as well as the rates at which the welfare increases with the market size differ across different mechanisms in a significant way. The efficiency of DACB rises quickly with market size, reaching 90% for $n = 1,000$, and above 96% for $n = 10,000$. By contrast, the efficiency ratio is fairly steady around 80-81% for DA, regardless of the size. As n rises in the range $1,000 \leq n \leq 10,000$, the DACB's gap relative to TTC narrows to 3%, while its gap relative to DA widens to 15%.

[24]The mechanisms were simulated under varying number of random drawings of the idiosyncratic and common utilities: $1000, 1000, 500, 500, 200, 200, 100, 100, 20, 10$ for the market sizes $n = 10, 20, 50, 100, 200, 500, 1,000, 2,000, 5,000, 10,000$.

This result shows that DACB performs well in efficiency even for relatively small markets. Figure 2 shows the fraction of blocking pairs under DA, DACB and TTC. Clearly, DA

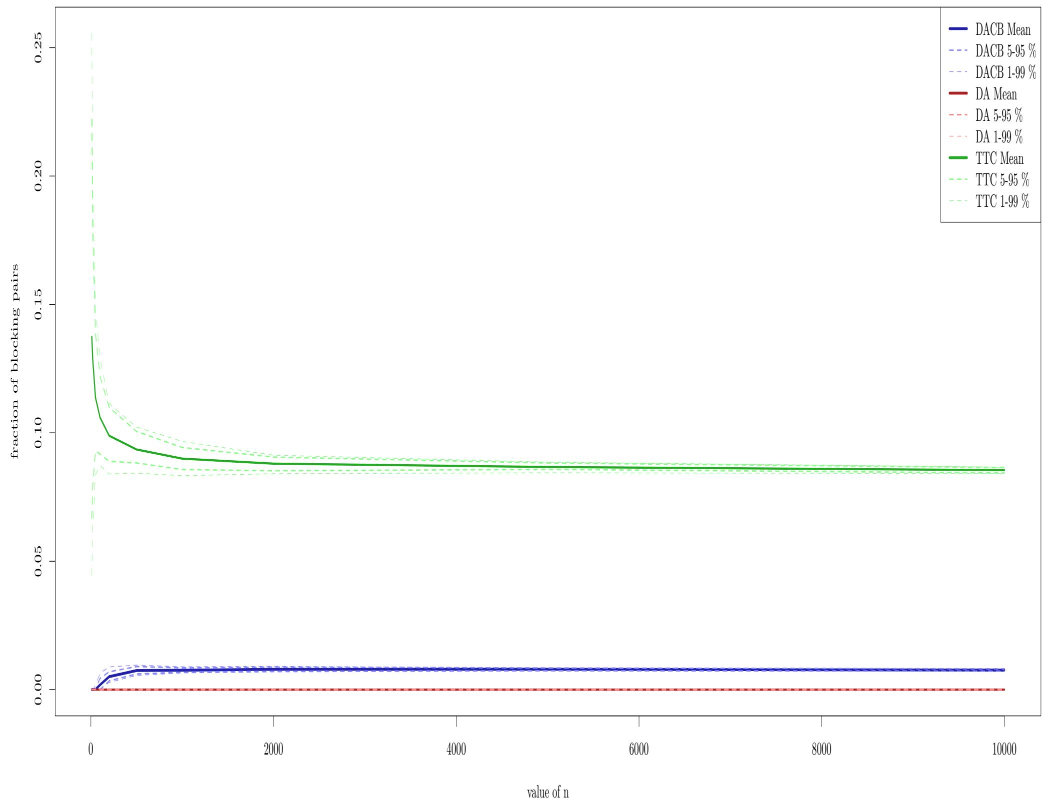


Figure 2: The fraction of blocking pairs under alternative mechanisms.

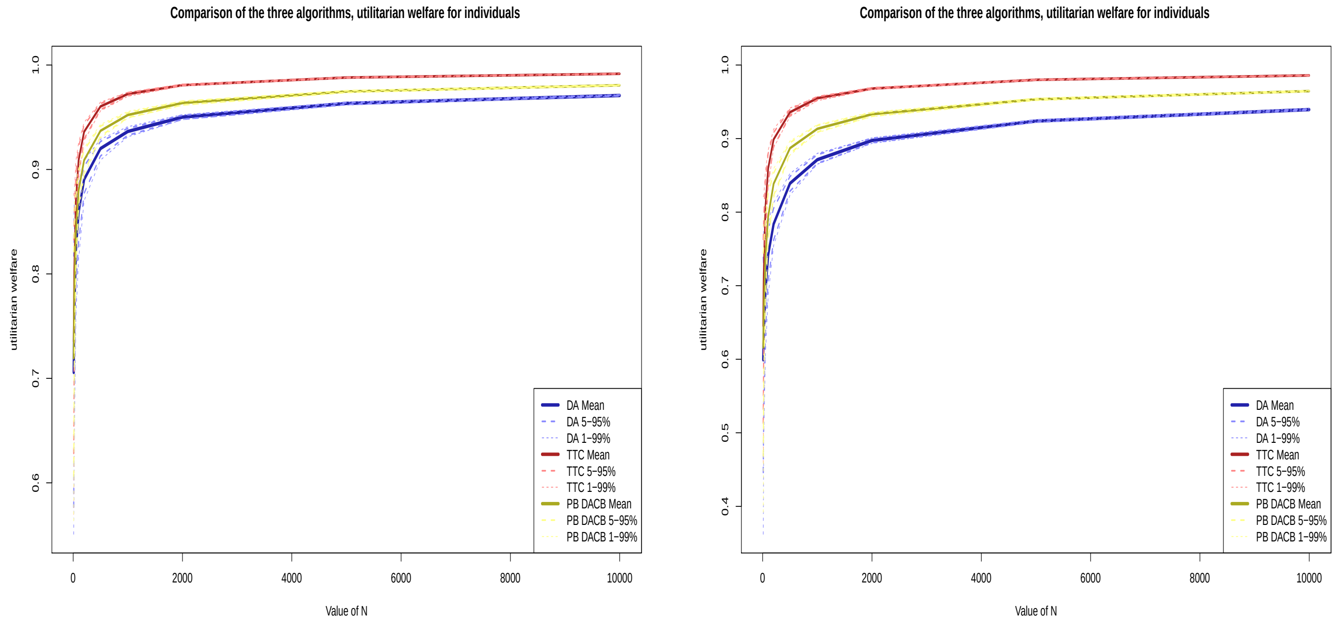
admits no blocking pairs, so the fraction is always zero. Between TTC and DACB, there is a substantial difference. Blocking pairs admitted by TTC comprise almost 9% of all possible pairs, whereas DACB admits blocking pairs that are less than 1% of all possible pairs, and these proportions do not vary much with the market size. These simulation results suggest that the DACB performs reasonably well even for moderate-size markets.

S.11.2 Robustness to general forms of preferences and priorities

Lee and Yariv (2017) show that any stable matching (and thus DA) is asymptotically efficient if agents' priorities have common component that is drawn from a non-degenerate interval according to an absolutely continuous distribution function *and* if the market is balanced (i.e., that the numbers of individuals and objects are the same). To see how DACB compares with DA under this environment, we performed several simulations under the assumption that agent i's priority at object o is given by $V(v_i, \eta_{i,o}) = v_i + \eta_{i,o}$, where both v_i and $\eta_{i,o}$ are each uniformly distributed from $[0, 1]$. We assume that agent's preference is given by $U(u_o, \xi_{i,o}) = u_o + \xi_{i,o}$ where $\xi_{i,o}$ is distributed uniformly from $[0, 1]$, and u_o is uniform on $[0, 1]$ or $[0, 3]$. When we run DACB in this environment, we set the serial orders of the agents according to their average priorities, i.e., the agent with the best average priority gets the first position in the serial order, the one with the second average priority gets the second, and so on. Hence, the name priority-based (PB) DACB. As discussed in Sections 5 and 6, this is how we envision DACB to be implemented in a pratical setting, namely with the serial order chosen to reflect the agents' priorities at objects. Finally, for the purpose of this simulation, we also set $\kappa(n) = \log^2(n)$ throughout.

Figure 3-(a) compares utilitarian welfare under DA, DACB and TTC when u_o is uniform on $[0, 1]$. As can be seen, the TTC performs best, followed by DACB, which performs in turn better than DA. While both DA and DACB attain higher welfare as the size of the market increases, the difference between the two is significant for a reasonable market size. Figure 3-(b) shows the same comparison when u_o is uniform on $[0, 3]$. This change simply means that agents' preferences exhibit higher correlation than before. Hence, as discussed in the text, this case implicitly involves market imbalance and excessive competition toward high quality objects. In light of our result in Section 4.2 and Ashlagi, Kanoria, and Leshno (2017), one would expect the gap between DACB and DA to widen in this case. This is indeed what we see here.

Last, Figures 4 compares the number of blocking pairs under alternative mechanisms in this environment. Compared with the case without correlation in agents' priorities (see Figure 2 in the main text), the fraction of blocking pairs under TTC is significantly lower (close to 3-6% as opposed to 10% in Figure 2 in the main text). The reason for this is that the agents assigned in early rounds of TTC tend to have high priorities even when they are assigned via long cycles, unlike the case of uncorrelated priorities. Nevertheless, the fraction of blocking pairs does not fall as the market grows large. This fact suggests that the asymptotic instability we find in Section 4.1 is robust to the introduction of correlated priorities. By contrast, the PB DACB eliminates almost all blocking pairs; the fraction of blocking pairs hovers around 1%.



(a) u_o uniform on $[0, 1]$

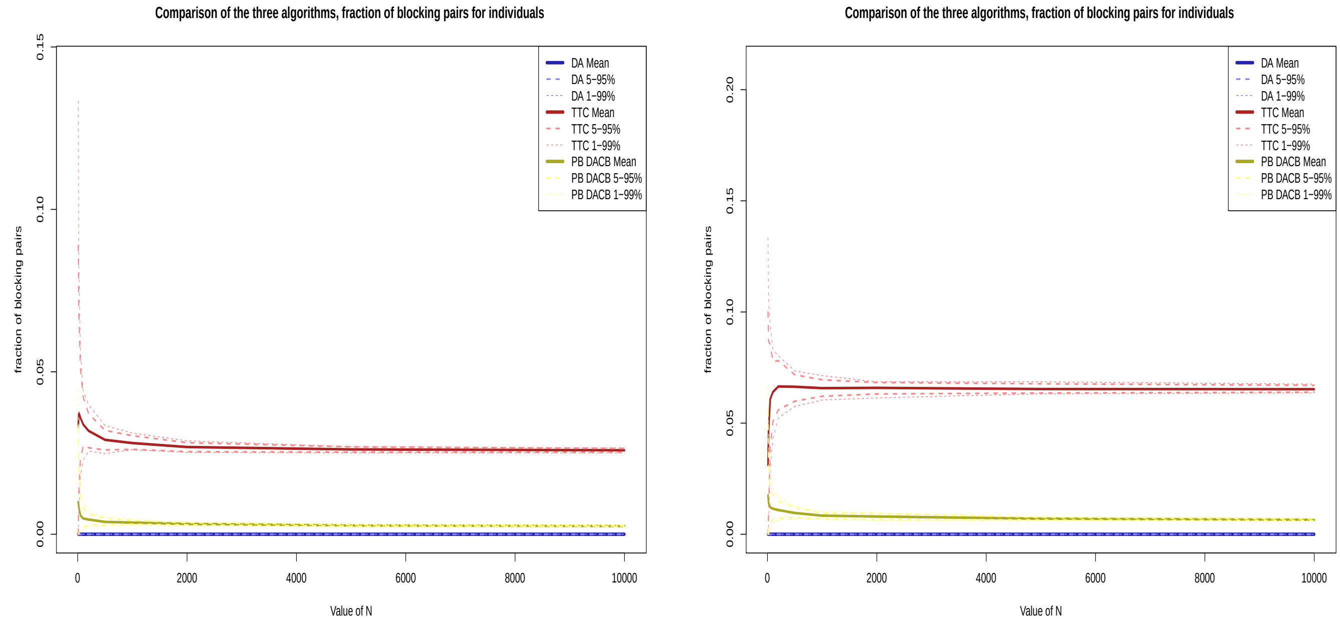
(b) u_o uniform on $[0, 3]$

Figure 3: Utilitarian welfare under correlated priorities.

S.11.3 The effect of preference correlation on mechanisms

Our results (Theorems 2 and 3) suggest that—under fairly general conditions—increased correlation in agents' preferences worsens the inefficiency of DA and the instability of TTC. However, Theorems 4 and 6 show that, asymptotically, DACB with appropriately chosen (κ, j) is efficient and stable. This last result remains true even when competition among agents is strong (for instance, with large common value differences across objects in different tiers). Our analytical results are, however, silent on how correlation affects the performance of DACB (for a fixed size of the economy). The simulations performed below show that, when competition increases due to preference correlation, the performance of DACB *declines in absolute terms* but its performance *improves in relative terms* compared with those of DA and TTC. Importantly, the simulations show how the choice of (κ, j) varies when correlation varies.

The simulations performed in this section use the very same setting as in Section S.11.1: $U(u_o, \xi_{i,o}) = u_o + \xi_{i,o}$ and $V(\eta_{i,o}) = \eta_{i,o}$, where $u_o, \xi_{i,o}$ and $\eta_{i,o}$ are distributed uniformly from $[0, 1]$. To increase correlation, we simply increase the size of the support of the distribution of u_o to $[0, 3]$. The size of the market is fixed at $n = 5,000$. Figure 5 depicts the "levels" of efficiency and stability achieved via DACB with various (κ, j)'s. Efficiency (the vertical axis) is measured by the percentage of agents who cannot be made better off



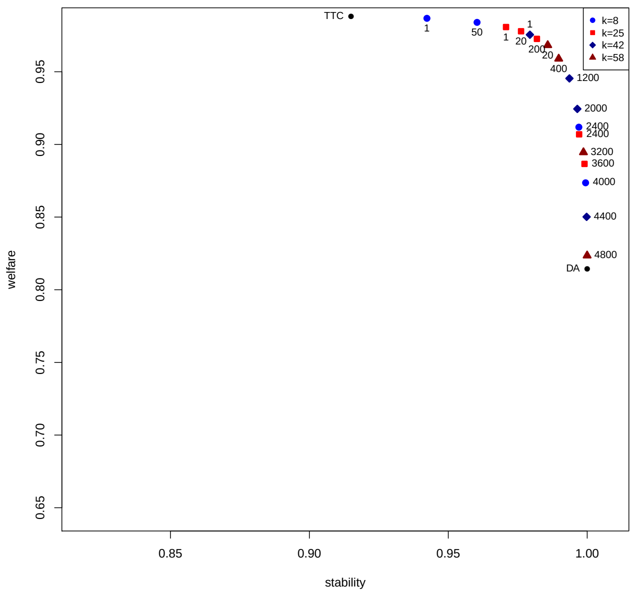
(a) u_o uniform on $[0, 1]$ (b) u_o uniform on $[0, 3]$

Figure 4: Number of blocking pairs under correlated priorities.

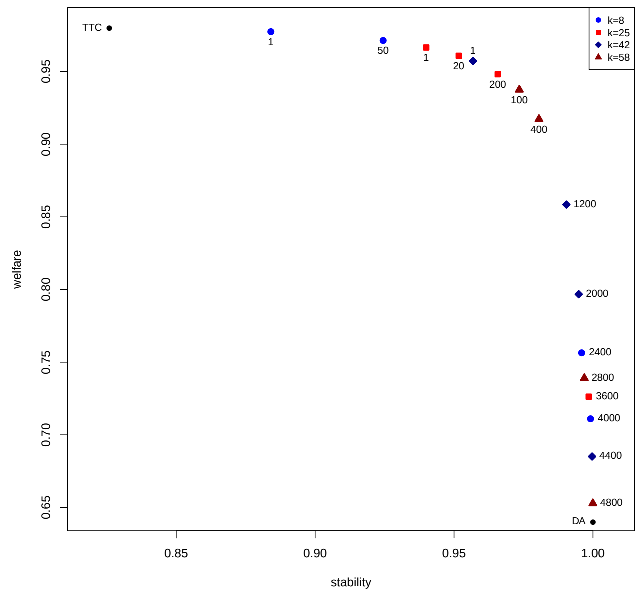
through Pareto-improving reallocation, while stability (the horizontal axis) is measured by the percentage of pairs (i, o) that do not form a block.[25] For both levels of correlation in agents' preferences (i.e., distribution of u_o) we plot the efficiency / stability performance of DACB for various values of (κ, j). We refer to the induced "frontier" as the efficiency / stability frontier.

We first note that allowing for a minor violation of stability (resp., efficiency) can significantly improve the efficiency (resp. stability) performance of our mechanism for both levels of correlation. For instance, for both distributions of u_o, tolerating 2% of blocking pairs allows one to reduce the efficiency loss by 15 to 20 percentage points compared to the situation where exact stability is imposed. Second, we observe that the efficiency / stability frontier is pushed inward after an increase in correlation among agents' preferences. More precisely, given any level of stability, the maximum level of efficiency one can achieve (on the frontier) decreases. This means that, for a fixed value of (κ, j), the absolute performance of DACB declines both in the efficiency and stability after an increase in correlation. However,

[25] Specifically, efficiency is defined as $1 - \frac{\#\text{ of Pareto-improvable agents}}{\#\text{ of total agents}}$ where the # of Pareto-improvable agents corresponds to the number of agents who are better off when running Shapley-Scarf TTC on top of the mechanism. As for stability, our percentage is defined by $1 - \frac{\#\text{ of blocking pairs}}{\#\text{ of possible pairs}}$ where we recall that the denominator is $n(n-1)$ for $n = 5{,}000$.



(a) u_o uniform on $[0, 1]$



(b) u_o uniform on $[0, 3]$

Figure 5: Efficiency vs. Stability.

Note: The shape of each coordinate corresponds to κ, while the associated integer refers to parameter j.

the relative performance of DACB over thoses of DA and TTC improves: after the increase in competition, the efficiency loss for DA is huge (almost 20 additional points) and the stability loss for TTC is also important (close to 10 additional points). For appropriate choices of (κ, j), the losses in the two dimensions for DACB can be limited to 3-4 points.

The simulation results also give us some guidance on how the choice of (κ, j) should change when competition increases. Again, increased correlation in agents' preferences reduces the performance of DACB in both efficiency and stability. However, the adverse effect on efficiency appears to become more severe than that on stability. For instance, focusing on the extreme points (DA and TTC), the efficiency measure is reduced by almost 20 points under DA while the stability loss is roughly of 8 points for TTC.[26] Hence, assuming that the designer has some fixed preference over the efficiency and stability trade-off, after an increase in correlation, the designer will find the outcome from the same (κ, j) to entail too much inefficiency compared with stability. Thus, such a designer will want to reduce the value of κ and/or j to restore what he considers to be the "optimal" values of efficiency and stability.

S.12 NYC Public High School Choice and the Associated Data Set

S.12.1 Institutional elements on how NYC actual system operates

School choice has a long history in New York City. For instance, selective high schools appeared in the early 20th schools. The current system established by the Bloomberg administration started in 2004 and is the most highly centralized that the city has ever known.[27] In this section, we only provide a brief description of the procedure and refer to Corcoran and Levin (2011) or Abdulkadiroglu, Pathak, and Roth (2009) for further descriptions. Each fall, about 90,000 students apply to enter a high school in New-York City in the following year. Most of them are 8th graders.[28] NYC has over 700 school

[26]Strictly speaking, TTC is not a particular case of DACB. However, serial dictatorship is a special case of DACB and gives almost the same outcome in terms of efficiency and stability as TTC in our simulations.

[27]It replaced a system where students could submit at most five choices outside of their zone or attend their zoned schools if any. Under this old system, schools offers were uncoordinated so that some students received multiple offers while a large fraction of students received no offers. After the main round of assignment more than a third of students were left with no assignment and ultimately got administratively assigned. See Abdulkadiroglu, Agarwal, and Pathak (2015).

[28]A small fraction of the students are 9th graders. Those are mainly coming from junior high schools including 9th grade.

programs.[29] There are seven types of “admission methods” which determine how a program orders students by priorities. As we will see, these priorities are often weak, causing a set of students to fall in the same priority class. In order to resolve these indeterminacies, a single tie-breaking rule is used: at the beginning of the process, a single random number is assigned to each student. Whenever necessary this number is used to break an indeterminacy in the priorities of a program. We now describe these seven admission methods.

Unscreened programs do not rank students at all, i.e., all students are in the same priority class. *Zoned* school programs give priority to students who live within a pre-defined zone area (some students can have no zoned program). For these two programs, there is no screening of students other than by lottery and geographic location. However, the other five programs, to some extent, evaluate students' abilities in some way or condition their priorities on interests applicants express in attending a school. For instance, for *limited unscreened* programs, all else being equal, students who attended an information session or visit the school's exhibit at high school fairs or open houses will have a higher priority. Similarly, *Screened* programs (all else equal) assign higher priority to students based on several criteria such as their 7th grade report card, reading and math standardized scores, attendance and punctuality, interview, essay or additional diagnostic tests. *Audition* programs partly base their priorities on auditions aimed at evaluating their proficiency in specific performing of visual arts, music, or dance. *Educational option* programs target a distribution in terms of reading ability measured by their score on the 7th grade standardized reading test. If possible, 16% of seats are assigned high performing, 68% middle performing students and the 16% remaining are assigned low performing students. Moreover, for half of their seats, educational option programs can actively rank students (while still respecting the target distribution) as screened schools do. For the other half, the random tie-breaking rule is used subject to the distributional constraint.[30] In addition, the system ensures admission to an educational option program for students within the top

[29] One special feature of NYC is that one school can offer several school programs. Indeed, there are about 400 high schools housing these programs. Students apply to programs, and the programs within the same high school can be very distinct. Hence, for our purpose, the relevant unit to focus on is the program.

[30] More precisely, each educational option program is split into six separate programs: LR, LS, MR, MS, HR, HS where L, M and H stand for low, middle or high achievement (in the reading test) while R and S stand for random and select. For a given educational option program, 50%×16% of the seats are for LR, 50%×68% are for MR and 50%×16% HR. Similarly, 50%×16% of the seats are for LS, 50%×68% are for MS and 50%×16% HS. For these “virtual subprograms”, high-level students are ranked above middle- and low-level students at HS/HR, middle-level students are ranked above high- and low-level students at MS/MR, and low-level students are ranked above high- and middle-level students at LS/LR. Within each class the selection process depends on whether the virtual subprogram is select (S) or random (R). For select subprograms, the selection within a category is similar to that of a screened programs. For random subprograms, the single random number is used to break ties.

2 percent on the seventh grade standardized reading test provided that this program is top ranked. Finally, *specialized high school* programs have a special status. Students willing to apply to these schools have to pass the Specialized High School Admission Test (SHSAT). Students' priorities in these schools are purely determined by their SHSAT score.[31]

In the summer following seventh grade, families are encouraged to review the Directory of the NYC Public High Schools published by the Department of Education. This document provides information on schools' locations, contact information, enrollment, academic performances along with details on how priorities are set. In the fall, students who are interested in entering a specialized high school program have to pass the SHSAT test. Programs requiring auditions and interviews conduct them during the fall semester as well.

Early December, these students participate in the *first round*. In that round, *all* students have to submit a list of up to twelve school programs by order of preferences. Students who passed the SHSAT test are allowed to submit an additional list of specialized school programs ordered by preferences.[32] The DA (student proposing) mechanism is used to produce the assignment for both specialized high schools and regular high school programs, separately.[33] In March, students receiving an offer from a specialized program (hence, who passed the SHSAT test) are informed of that offer together with the additional offer (if any) they may have received from a non-specialized program. They are asked to pick one of the two offers.

Students who did not get any offer in round 1 or did not participate in round 1 go through a *second round* (sometimes referred to as the main round).[34] Capacities of schools are adjusted based on the decisions made by the students assigned in the first round. The same algorithm is used to assign students, using the ROLs these students had submitted early on. The vast majority of students is assigned in that round. However, if a student is unassigned, there is a third round in April where this student can again submit a new set of up to twelve choices among remaining school programs. In this third round DA is again used to assign students. In case a student is still unmatched he will be assigned a school as close as possible to his residence. In addition, if there are sufficient grounds, a student may appeal in May. Finally, students who were not present for this high school admission process have to meet with an admissions counselor at the enrollment office in order to get

[31]Only a small fraction of seats at these schools are opened to disadvantaged students who performed well at the SHSAT but who were below the cut-off score for acceptance.

[32]There are nine specialized high school programs in NYC.

[33]To handle the target distribution of Educational Option programs (see footnote 30), the rank order list of a student who applies to such a program is modified to rank the six "virtual" subprograms according to the order HR, HS, MR, MS, LR, LS.

[34]See http://schools.nyc.gov/ChoicesEnrollment/High/events/default.htm for details on the second round application process.

assigned a high school. This last round is usually referred to as over-the-counter.[35]

S.12.2 Data sets

The NYC Department of Education (DOE) provided us with several data sets. We used data for academic year 2009-2010 which for each round (round 1, 2 and 3), contains the ROLs of the students who participated in that round as well as the assignments achieved at the end of that round. In addition to this, for each school program that a student ranks, the data contains information on the priority of the student at that school. Thus, we can reconstruct each school program's priorities (at least over students who ranked that program) – see next section for additional details.

Removal of data points. We removed from our data set the small number of students for whom we had missing information such as ZIP code or information on the average income in their ZIP codes. Students with ethnicities different of Black, Hispanic, Asian or White were also removed.[36] We deleted programs for which we had missing information, in particular, those which do not appear in the NYC high school directories that we use to reconstruct priorities (see below). We also removed all programs which closed by the end of the year (students's behavior suggests that they were aware that these school programs were about to close). In addition, as pointed out by Abdulkadiroglu, Pathak, and Roth (2009), the system does not give incentives to students who scored within the top 2 percent on the seventh grade standardized reading test to report truthfully their preferences. Hence, we removed these students.

Data for preference estimation. For the estimation of preferences, we used students participating in Round 1 and 2 (there is no additional student participating in Round 3).[37] Eventually, we had 70,559 students and 694 school programs available for preference estimation.

Data for Comparison of Mechanisms. To run our mechanisms, we restricted attention to students being matched to a program either in Round 2 or in Round 3. We also deleted from the student's ROL the program if the student is not eligible.[38] Finally, the

[35]The number of such students is surprisingly large (around 36,000 every year). These are among the school system's highest-needs students. See Arvidsson, Fruchter, and Mokhtar (2013).

[36]Ie., students with Multi-Racial/Mixed Ethnicity as well as American Indian or Alaskan Natives.

[37]Recall that students participating in Round 1 submit a list on non-exam schools.

[38]A student is not eligible at a school if he/she does not meet at least one of the admission priorities listed in the NYC high school directory for that school (i.e., if he has no priority group) or if the school

information about each program's capacity is not available. In order to estimate the capacities of a school, we used the enrollments through Rounds 2 and 3 at that school. Since we only kept those students who are matched to a school either at Round 2 or Round 3, by construction, the number of students involved is exactly equal to the total number of seats at schools.[39] We eventually ended up having 66,392 students and 694 programs as inputs for our algorithms.

S.12.3 Strategies to complete missing data

Priorities. For each school program that a student ranks, the student is assigned a *priority group* and a *priority rank*. These two numbers are meant to allow us to reconstruct each school program's priorities. Priority group specifies coarse equivalent classes. Students with small priority groups numbers always have higher priority than students with larger ones. Within each coarse priority class so defined, whenever available, the priority rank number is used to further discriminate between students' priorities. This lexicographic order between priority group and priority rank is (based on the explanation by Department of Education staff) the alleged procedure.[40]

As mentioned in the paper, an applicant's priority is observed in the current data set only for the programs the applicant lists in his/her ROL. For our purposes, we need to reconstruct these priorities for programs that an applicant does not list. There are two reasons why we need to reconstruct the missing priority information. First, when we compare mechanisms based on observed ROLs (Section 6.1 in the paper), the missing priority information for programs not listed in ROL presents some issues with calibrating TTC. Indeed, under standard TTC, a student may be able to trade a seat of a school at which he has a high priority – even though he may not list that school – with a seat at

did not assign him any priority rank.

[39] Abdulkadiroglu, Pathak, and Roth (2009) use the final fall enrollments as estimates of capacities. However, these figures include a sizable number of students who are admitted through the over-the-counter round. Evidently, the seats for these applicants are created during this administrative assignment process and they are not available during Round 2 assignment; many programs who reject some applicants in Round 2 end up admitting students in administrative assignment. Given this, the fall enrollments would over-estimate the capacities used for the Round 2 assignment. We believe that the qualitative nature of the comparison of alternative matching algorithms performed here and also in Abdulkadiroglu, Pathak, and Roth (2009) is robust to alternative methods of estimating capacities.

[40] We do note, however, that there are incidences of these lexicographic rule being violated. The incidences comprise a rather small fraction. For instance, there are 2645 students who see their priority group number violated, i.e. who are rejected from a school program while others with higher priority group number are accepted; and 4051 students who see their priority group number violated or are rejected from a school program while others with a same priority group number and a higher priority rank are accepted. In any case, the priority rule serves our purpose, which is to consider a realistic market setting.

a school that he likes (but where his priority may be low).[41] Second, when we compare mechanisms based on structural estimates (Section 6.2 in the paper), the completed ROLs contain schools which are not in the observed ROLs. However, to run our mechanisms, we do need to get the priorities of students at these schools.

To reconstruct the missing priorities we proceed as follows. First, the rules used at a school to assign priority groups to students are publicly available (in NYC public high school directories). These rules are function of students' characteristics that we observe in our data and so we can reconstruct priority groups of students at schools which do not appear in the observed ROL.[42]

As we already explained, priority rank obeys rules that are not fully explicit. So to reconstruct priority ranks, we proceeded in two steps. First, for a school program giving priority ranks to students (audition, educational option, and screen programs) we estimate the probability that a student gets a rank. Indeed, some students do not get any priority rank meaning that these students are unacceptable at the school. In order to estimate this probability, we focus on the students who listed the program, and we run a logistic stepwise regression (combining backward elimination and forward selection of explanatory variables).[43] In a second step, we estimate the distribution over priority ranks of a student given that he does receive a priority rank. More specifically, focusing on the students who got a priority rank at the program, we run a linear stepwise regression (again, combining backward elimination and forward selection within the same explanatory variables) in order to estimate the priority rank of students.

Eventually, for each program (audition, educational option, or screened) and for each student who did not list that program in his/her observed ROL, we obtain (1) the probability that this student gets a rank at that program and (2) the probability distribution of getting alternative ranks conditional on getting a priority rank.

Geographic distance. Recall that we do not have the street addresses for students but only the ZIP code of their residence. We used google maps API (and library ggmap in R) to geocode the addresses of the high schools into spatial coordinates and to geocode the

[41]However, it does not cause any issue for the implementation of DA or DACB (the way the priorities of students are set at schools that do not appear in their list has no impact on the final assignment).

[42]Many schools have rules referring to the district the student is studying at or living at. While we have information on the district where individuals study, we do not have the exact students' addresses (we only have the ZIP codes). Hence, in some cases, we ignored the rule referring to the district where the student lives.

[43]Our explanatory variables are based on the following student characteristics: scores of the standardized Math test and the standardized English test for middle schoolers, the number of days of absence in the previous school year and the number of days with class lates in the previous year. The standardized scores of each of these variables constitute our set of explanatory variables.

ZIP code of the student into the spatial coordinates of its centroid. We used the library Imap in R to compute the geodesic distance. The geodesic distance between two points is specified by latitude/longitude using Vincenty inverse formula for ellipsoid. The distances are expressed in miles.

S.13 Preference Estimates

Below we report the posterior means and standard deviations of the estimated parameters obtained by the Gibbs sampling procedure. Estimates are made with submitted ranks of 70,559 students over 694 program choices.

High Math Achievement	
Main Effect	-0.048^{**}
Baseline Math	0.064^{***}
Baseline English	0.034^{***}
Subsidized Lunch	0.007^{**}
Neighborhood Income	0.012^{***}
Limited English Proficient	0.018^{***}
Special Education	-0.033^{***}
Percent Subsidized Lunch	
Main Effect	-0.020^{**}
Asian	-0.021^{***}
Black	0.003^{**}
Hisp	0.031^{***}
Subsidized Lunch	0.029^{***}
Neighborhood Income in (10000s)	-0.008^{***}
Size of 9th Grade	
Main Effect	-0.201^{***}
Baseline Math	0.043^{***}
Baseline English	-0.031^{***}
Subsidized Lunch	0.108^{***}
Neighborhood Income G9	0.033^{***}
Special Education	0.006
Percent White	
Main Effect	0.106^{***}
Asian	-0.086^{***}
Black	-0.149^{***}
Hisp	-0.096^{***}

Spanish Language Program	
Limited English Proficiency	−0.398
Limited English Proficiency x Hisp	6.399***
Asian Language Program	
Limited English Proficiency	2.297***
Limited English Proficiency x Asian	6.393***
Other Language Program	
Limited English Proficiency	5.609***
Standard Deviation of ϵ	25.78***
Standard Deviation of ξ	5.66***
Random Coefficients (Covariances)	
Percent White : High Math Achievement	0.001***
Percent White : Percent Subsidized Lunch	0.001***
Percent White : Size of 9th Grade	0.000***
Percent White : Percent White	0.006***
Size of 9th Grade : High Math Achievement	0.000
Size of 9th Grade : Percent Subsidized Lunch	0.007***
Size of 9th Grade : Size of 9th Grade	0.132***
Percent Subsidized Lunch : High Math Achievement	0.000*
Percent Subsidized Lunch : Percent Subsidized Lunch	0.004***
High Math Achievement : High Math Achievement	0.015***
Number of Students	70559
Number of Ranks	468163

Note: *** 0.001% significance ** 0.01% significance * 0.05% significance · 0.1% significance

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