Available online at www.sciencedirect.com

SciVerse ScienceDirect

Journal of Economic Theory 147 (2012) 1140–1164

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

Log-linear dynamics and local potential

Daijiro Okada, Olivier Tercieux *

Paris School of Economics and CNRS, Paris, France

Received 28 October 2010; final version received 7 October 2011; accepted 20 November 2011

Available online 17 February 2012

Abstract

We show that local potential maximizer (Morris and Ui (2005) [14]), a generalization of potential maximizer, is stochastically stable in the log-linear dynamic if the payoff functions are, or the associated local potential is, supermodular. Thus an equilibrium selection result similar to those on robustness to incomplete information (Morris and Ui (2005) [14]), and on perfect foresight dynamic (Oyama et al. (2008) [18]) holds for the log-linear dynamic. An example shows that stochastic stability of an LP-max is not guaranteed for non-potential games without the supermodularity condition. We investigate sensitivity of the log-linear dynamic to cardinal payoffs and its consequence on the stability of weighted local potential maximizer. In particular, for 2×2 games, we examine a modified log-linear dynamic (*relative log-linear dynamic*) under which local potential maximizer with positive weights is stochastically stable. The proof of the main result relies on an elementary method for stochastic ordering of Markov chains.

JEL classification: C72; C73

Keywords: Log-linear dynamic; Relative log-linear dynamic; Stochastic stability; Local potential maximizer; Equilibrium selection; Stochastic order; Comparison of Markov chains

1. Introduction

Game theorists have developed dynamic processes based on specific behavioral rules in order to identify solutions to games, if any, that are stable in the long run. A well-known class of dynamics studied in this field is the perturbed best-response processes in which each player chooses

* Corresponding author.
E-mail address: tercieux@pse.ens.fr (O. Tercieux).

0022-0531/$ – see front matter
doi:10.1016/j.jet.2012.01.011

a non-best response, a mistake, to the current strategy profile with a small probability (e.g., [9, 25]).[1] Within this framework, it has been demonstrated (see [2]) that any Nash equilibrium can be made stable by a suitable choice of mistake probabilities provided that they are independent of the current play. Hence the necessity to have a coherent theory on the way mistakes occur.

In the log-linear dynamic [3,4], costlier mistakes are less, exponentially less to be precise, likely to occur (cf. proper equilibrium concept in [16]). A recent work [1] identifies a formula for the stochastic potential (see [6,25]) for this dynamic. In principle, the long-run stable state can be identified as those that minimize the stochastic potential. In contrast and complement to this approach, one may wish to identify the nature of equilibria (e.g., Pareto-dominance, risk-dominance) that depends only on the underlying game, and ensures long-run stability.[2] For potential games (see [12]), the long-run stable states of the log-linear dynamic are those strategy profiles that maximize the potential function (see [3,4]).

In this article, we consider the log-linear dynamic on games that admit a local potential function, a generalization of potential function (see [13,14]). We provide a necessary and sufficient condition for a game to belong to this class as well as a formula for a local potential function. This condition refers only to the payoff functions of the game and is computationally easy to verify. Our main result is that the long-run stable states for such games are the strategy profiles that maximize the local potential function if the payoff functions are supermodular or the local potential function is. This result holds under a wide range of ways players may interact: fixed set of players, population (or random matching) games, and more general forms of local interactions (see [4]).

Our main result has several counterparts in the literature on equilibrium selection and robustness of equilibrium. Monotonicity conditions similar to our supermodularity routinely appear in papers that use the local potential ideas to go beyond results obtained for potential games (e.g., [5,14,18,19]). Whether these selection results hold without such conditions, and whether results in these papers can be improved upon to cover all potential games, have been open questions in all these works. We present examples that provide clear answers to these questions in the context of the log-linear dynamic. Specifically, we demonstrate that (a) stability of an LP-max is not guaranteed for non-potential games without the supermodularity condition, and, (b) there are some potential games outside applicability of our main result.

On the one hand, our result is complementary to the existing result on potential games. The latter covers all, but only, potential games while our result covers some potential games and non-potential games. On the other hand, our result has an advantage in that it identifies a stochastically stable state that is robust to perturbations. Since potential games are non-generic, a result for them need not apply to (arbitrarily) nearby games. In contrast, there are open set of games which admit a local potential function, and satisfy the sufficient condition of our theorem.

We also discuss possibilities for, and obstacles to, the stability of weighted potential games and the corresponding generalization to games with weighted local potential functions. This discussion highlights in two ways the nature of the log-linear dynamic, i.e., its sensitivity to the cardinality of payoffs. First, weighted potential maximizer need not be stochastically stable under the standard log-linear dynamic.[3] Second we consider a variant of log-linear dynamic

[1] The best response dynamic is a special case of the Darwinian process, studied in [9], according to which a better action is better represented in the next generation.

[2] Computation of stochastic potential depends on the specific parameters of the dynamic process as well as on the game being played. Also, minimization of stochastic potential is computationally difficult (see [25]).

[3] See Example 5 (due to [1]).

(relative log-linear dynamic), applicable to 2-by-2 games, that depends on the relative magnitude of payoffs. We show that strategy profiles that maximize a weighted local potential function with positive weights are stochastically stable under this dynamic.

Finally, let us provide a basic idea behind the proof of the main theorem. We will offer more detailed sketch at the end of Section 4 before we present the full proof in Section 6. Recall that, for a potential game, the difference in payoffs to any one player due to his unilateral deviation is equal to the difference in the values of a common function (potential function) at these profiles. The concept of local potential replaces the equality in this definition with an inequality. Thus games we consider are the ones in which the payoff difference is bounded by the corresponding difference in the values of a common function, called a local potential, rather than potential, function. You may say that such game is "bounded" by a potential game. The basic insight for the proof is that the dynamic for a game with a local potential function may be similarly bounded in some sense by the dynamic for a potential game.

In the log-linear dynamic, an action that generates a larger (positive) payoff difference over the current payoff is more likely to be chosen. For a potential game, this dynamic moves toward a strategy profile that maximizes the potential function (see [3,4]). If the underlying game is not an exact potential game but is bounded by a potential game in the sense discussed above, then the same probabilistic tendency towards a (local) potential maximizer may be at work. In a sense, the dynamic for the potential game nudges the dynamic for the local potential game towards the same long-run stable states. Technically, we obtain our main result by ordering two log-linear dynamics (Markov chains), one for the original game and the other for the potential game that bounds it. Our proof employs only elementary tools of finite Markov chains and stochastic dominance.

The paper is organized as follows. The basic setup is laid out in Section 2. The central concept of local potential is defined in Section 3 where we also provide a necessary and sufficient condition for a strategy profile to be a local potential maximizer. We present our main result (Theorem 2) as well as a sketch of its proof in Section 4. In this section we also present an example illustrating the indispensability of supermodularity condition in our main theorem, when applied to non-potential games, as well as an example highlighting the difference between our main theorem and the existing result on potential games. In Section 5, we discuss the stability of weighted local potential maximizers (see [14]) and how this relates to the sensitivity of the log-linear dynamic to affine transformations. In particular, we study a modified dynamic, called relative log-linear dynamic, that is invariant to affine transformations of payoffs. The proof of the main result is presented in Section 6. In Section 7, we present a local interaction version of the log-linear dynamic and the corresponding results. Section 8 concludes the paper.

2. The basic model

Consider an I-person finite game in strategic form. The set of actions available to player i is S_i and his payoff function is $u_i : S \to \mathbb{R}$ where $S = S_1 \times \cdots \times S_I$. Symbols such as s_{-i} will be used with the usual meaning.

The dynamic under consideration runs in discrete time $t = 0, 1, 2, \ldots$ and its state space is S. At $t = 0$ a strategy profile is selected according to an initial distribution. At each subsequent period a single player is selected[4] and is given an opportunity to revise her strategy according to

[4] See [10] for variations on this and other structural aspects of log-linear dynamics.

a stochastic choice rule. The probability that player i is given this opportunity is denoted by ρ_i. We assume $\rho_i > 0$ for all i. Thus state can change from s to s' if, and only if, $s' = (s'_i, s_{-i})$ for some i and $s'_i \in S_i$.

In this article we study the log-linear stochastic choice rule according to which the log likelihood ratio between two actions is proportional to the difference between the payoffs from these actions. The factor of proportionality, a nonnegative real number, is denoted by β. Let $p_i(s_i \mid s : u_i, \beta)$ be the probability that player i chooses $s_i \in S_i$ given a state $s \in S$. The log-linear stochastic choice rule is characterized by

$$\ln \frac{p_i(s''_i \mid s : u_i, \beta)}{p_i(s'_i \mid s : u_i, \beta)} = \beta\big(u_i\big(s''_i, s_{-i}\big) - u_i\big(s'_i, s_{-i}\big)\big) \tag{2.1}$$

for all $s \in S$ and $s'_i, s''_i \in S_i$. Thus, given a revision opportunity and a current state s, player i is exponentially more likely to select s''_i than s'_i whenever s''_i is a better reply to s_{-i} than s'_i is. Equivalently,

$$p_i\big(s'_i \bigm| s : u_i, \beta\big) = \frac{e^{\beta u_i(s'_i, s_{-i})}}{\sum_{s''_i \in S_i} e^{\beta u_i(s''_i, s_{-i})}}. \tag{2.2}$$

It is clear from (2.1) and (2.2) that the log-linear rule $p_i(\cdot \mid s : u_i, \beta)$ is simply the uniform distribution on S_i when $\beta = 0$, and it converges as $\beta \to \infty$ to the uniform distribution over the best responses against s_{-i}. For this reason the log-linear dynamic is considered a perturbation of the best-response dynamic, with an explicit rationale for the way non-best-response actions might be chosen. Note that we have taken β to be common to all players. We will discuss a dynamic with β's varying across players in Section 5.

The log-linear choice rules, $p_1, \dots, p_I$, generate a (time-homogeneous) Markov chain on the set of strategy profiles S with the transition probability from s to s' given by

$$q_{ss'}(u, \beta) = \sum_{i=1}^{I} \mathbf{I}\big(s'_{-i} = s_{-i}\big)\rho_i p_i\big(s'_i \bigm| s : u_i, \beta\big) \tag{2.3}$$

where $u = (u_1, \dots, u_I)$, and $\mathbf{I}$ is the indicator function. Let $Q(u, \beta) = (q_{ss'}(u, \beta))_{s,s' \in S}$ be the resulting transition matrix. The transition matrix when each player uses the best response rule is denoted by $Q^*(u)$. It follows from the comment after (2.2) that $Q(u, \beta) \underset{\beta \to \infty}{\longrightarrow} Q^*(u)$.

It is straightforward to see that the Markov chain associated with $Q(u, \beta)$ is irreducible. Hence it has a unique invariant distribution $\mu(u, \beta) = (\mu_s(u, \beta))_{s \in S}$, i.e., the unique solution to $\mu Q(u, \beta) = \mu$, and $(I + Q(u, \beta) + \cdots + Q(u, \beta)^t)/(t + 1)$ converges as $t \to \infty$ to a matrix whose rows are identical to $\mu(u, \beta)$. So $\mu_s(u, \beta)$ is also the asymptotic average frequency with which state s is visited. In addition, it is easy to verify that this chain is aperiodic and hence $Q(u, \beta)^t$ also converges to the matrix with all rows equal to $\mu(u, \beta)$. In contrast, the chain associated with $Q^*(u)$ typically has multiple recurrent classes and more than one invariant distribution. A well-known result states that $\lim_{\beta \to \infty} \mu(u, \beta)$ exists and is an invariant distribution of the chain associated with $Q^*(u)$ (e.g., [25]).

Definition 1. A state $s \in S$ is stochastically stable if $\lim_{\beta \to \infty} \mu_s(u, \beta) > 0$.

Thus states that are not stochastically stable will be observed with a vanishing frequency in the long run under any log-linear dynamic that is sufficiently close to the best response dynamic.

3. Local potential

In this section we discuss the main concepts used in this paper: local potential function and local potential maximizer. It generalizes the concept of potential function and potential maximizer (see [12]). Recall that a strategy profile s^* in a game $(S_i, u_i)_{i=1,\dots,I}$ is a potential maximizer[5] if there exists a function (potential function) $v : S \to \mathbb{R}$ with the property that s^* maximizes v and

$$u_i\big(s_i', s_{-i}\big) - u_i(s_i, s_{-i}) = v\big(s_i', s_{-i}\big) - v(s_i, s_{-i}) \tag{3.1}$$

for all $i = 1, \dots, I$, $s_i, s_i' \in S_i$ and $s_{-i} \in S_{-i}$.

For a game where every player's payoff function is identically v, let $q_{ss'}(v, \beta)$ and $Q(v, \beta)$ be the transition probability and the transition matrix for the corresponding log-linear dynamic. The invariant distribution for $Q(v, \beta)$ is denoted by $\mu(v, \beta)$. If a game with payoff functions $u = (u_1, \dots, u_I)$ has a potential function v, then $Q(u, \beta) = Q(v, \beta)$ and $\mu(u, \beta) = \mu(v, \beta)$. This is an immediate consequence of (3.1). The log-linear dynamic for a potential game is a reversible Markov chain and its invariant distribution can be explicitly obtained by solving the detailed balance conditions, $\mu_s q_{ss'}(v, \beta) = \mu_{s'} q_{s's}(v, \beta)$ for each $s, s' \in S$. Consequently, the set of stochastically stable states can be explicitly characterized.[6]

Theorem 1. *(See [3,4].) Suppose that $(S_i, u_i)_{i=1,\dots,I}$ is a potential game with a potential function v. Then the invariant distribution of $Q(v, \beta)$ is*

$$\mu_s(v, \beta) = \frac{e^{\beta\, v(s)}}{\sum_{s' \in S} e^{\beta\, v(s')}} \tag{3.2}$$

and $s \in S$ is stochastically stable if, and only if, s maximizes v.

The concept of local potential is obtained by relaxing (3.1) in two ways: equality is replaced by an inequality, and requiring the relationship between u_i and v to hold only "locally" with respect to some order of the strategies.

Definition 2. (See [13].) A strategy profile $s^* = (s_1^*, \dots, s_I^*)$ is a *local potential maximizer* (LP-max) for payoff functions $u = (u_1, \dots, u_I)$ if there exist

(1) a total order $\leqslant_i$ on each S_i, $i = 1, \dots, I$,
(2) a function $v : S \to \mathbb{R}$ (local potential function) such that
  (a) $v(s^*) > v(s)$ for all $s \neq s^*$,
  (b) for each i, and every $s_{-i} \in S_{-i}$,
    (b-1) if $s_i \leqslant_i s_i^*$ and $s_i \neq s_i^*$, then

$$v\big(s_i^+, s_{-i}\big) - v(s_i, s_{-i}) \leqslant u_i\big(s_i^+, s_{-i}\big) - u_i(s_i, s_{-i}),$$

[5] Defining a maximizer as a primary concept makes it easier to understand the difference between the potential and the local potential concepts in Definition 2.

[6] A potential maximizer is also stochastically stable under a range of modifications to the log-linear dynamic: the way revision opportunity is given to (a set of) players, restrictions on the set of actions available to revising players, and so on. See [10]. Our main result, stability of local potential maximizer, holds under the same alternative specifications.

(b-2) if $s_i^* \leqslant_i s_i$ and $s_i \neq s_i^*$, then

$$v(s_i, s_{-i}) - v(s_i^-, s_{-i}) \geqslant u_i(s_i, s_{-i}) - u_i(s_i^-, s_{-i}),$$

where s_i^+ (resp. s_i^-) is the smallest (resp. largest) element of S_i that is larger (resp. smaller) than s_i.

A few remarks on this definition are in order. First, Definition 2 (2)-(a) requires s^* to be the unique maximizer of the function v. If s^* is the unique maximizer of a potential function, then all requirements in Definition 2-(2) are met for any orderings of actions.

Second, a local potential maximizer as a set-valued concept is defined in [14], generalizing Definition 2 and the case when there are multiple potential maximizers. In the set-valued version, a local potential function is defined to be a measurable function on the set of action profiles S endowed with an algebra whose elements are partially ordered. A local potential maximizer is then defined to be the unique element of the algebra on which the local potential function attains the maximum. All of our results hold under the set-valued notion of local potential maximizer. See Appendix B and [17] for the details. In this article we present results and examples for a special case of Definition 2 as it will convey the most important substance of this work without extraneous technical details.

Finally, in the definition of [13], the right hand side of the inequalities in (b) are weighted by a nonnegative constant $w_i(s_i)$. Thus, definition we employ here is stronger, just as a potential function is a special case of a weighted potential function (see [12]). We discuss the weighted version of a LP-max in Section 5.

From Definition 2 it may seem difficult to see whether a given game admits an LP-max or how to find a local potential function.[7] Below we provide a necessary and sufficient condition for a strategy profile to be an LP-max. The condition is easy to check and also yields a formula for local potential function.[8]

To explain the condition in words first, let us consider the case where each component of a candidate LP-max, s_i^*, is the largest element in the given order on S_i. The necessary and sufficient condition for $s^* = (s_1^*, \dots, s_I^*)$ to be an LP-max is as follows. Take any other strategy profile s and consider any sequence of strategy profiles starting at s^*, ending at s, and at each step one, and only one, player deviates to a strategy that is lower in the given order than the previous one. If the sum of payoff differences for the deviating players along any such path is strictly negative, then, and only then, s^* is an LP-max.

Fix a total order $\leqslant_i$ on each S_i. We say that a finite sequence of strategy profiles, $s^0, s^1, \dots, s^L$, is a *monotonic path of unilateral deviations* if

(i) for each $\ell = 1, \dots, L$, there is one, and only one, player i_ℓ such that $s_{i_\ell}^\ell \neq s_{i_\ell}^{\ell-1}$,

and

(ii) for each player i, either $s_i^0 \leqslant_i s_i^1 \leqslant_i \cdots \leqslant_i s_i^L$ or $s_i^L \leqslant_i s_i^{L-1} \leqslant_i \cdots \leqslant_i s_i^0$.

[7] Though, given total orders on strategy sets and a candidate for an LP-max, the problem of finding a local potential function is that of linear programming.

[8] It is similar to the conditions for a game to be a potential game (see [12]).

For each monotonic path of unilateral deviations $s^0, s^1, \dots, s^L$ let

$$\Lambda(s^0, s^1, \dots, s^L) = \sum_{\ell=1}^{L} \big[u_{i_\ell}(s^\ell) - u_{i_\ell}(s^{\ell-1})\big], \tag{3.3}$$

which is the sum of payoff differences for the deviating players along the path.

Proposition 1. *A strategy profile s^* is a local potential maximizer for $u = (u_1, \dots, u_I)$ if, and only if, there is a total order $\leqslant_i$ on S_i for each $i = 1, \dots, I$ such that $\Lambda(s^0, s^1, \dots, s^L) < 0$ for every monotonic path of unilateral deviations $s^0 = s^*, s^1, \dots, s^L \neq s^*$. In addition, under this condition, the function $v : S \to \mathbb{R}$ defined by the formula below serves as a local potential function*:

$$v(s) = \begin{cases} 0 & \text{if } s = s^*, \\ \max \Lambda(s^0, \dots, s^L) & \text{if } s \neq s^*, \end{cases} \tag{3.4}$$

where max *is taken over all monotonic paths of unilateral deviations starting at $s^0 = s^*$ and ending at $s^L = s$.*

The proof of this proposition is given in Appendix A.[9] Let us apply Proposition 1 to a version of unanimity game (cf. [14] and [18]). This is not a potential game thus it provides a nontrivial use of the proposition.

Example 1. Each player has two actions, $S_i = \{0, 1\}$. Let $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$. The payoffs are such that $u_i(\mathbf{0}) > 0$, $u_i(\mathbf{1}) > 0$ and $u_i(s) = 0$ for all $s \neq \mathbf{0}, \mathbf{1}$. Thus there are two strict equilibria, $\mathbf{0}$ and $\mathbf{1}$. Let us show that the strategy profile $\mathbf{1}$ (resp. $\mathbf{0}$) is an LP-max if, and only if, $u_i(\mathbf{1}) > u_j(\mathbf{0})$ (resp. $u_i(\mathbf{0}) > u_j(\mathbf{1})$) for all i and $j \neq i$.

Let $0 \leqslant_i 1$ for $i = 1, 2$. Pick an individually monotonic path of unilateral deviations $s^0 = \mathbf{1}, s^1, \dots, s^L \neq \mathbf{1}$. Clearly, $u_{i_1}(s^1) - u_{i_1}(s^0) = -u_{i_1}(\mathbf{1})$, $u_{i_\ell}(s^\ell) - u_{i_\ell}(s^{\ell-1}) = 0$ for all $\ell = 2, \dots, L-1$, and $u_{i_L}(s^L) - u_{i_L}(s^{L-1}) = u_{i_L}(\mathbf{0})$ (if $s^L = \mathbf{0}$) or $u_{i_L}(s^L) - u_{i_L}(s^{L-1}) = 0$ (if $s^L \neq \mathbf{0}$). Hence $\Lambda(s^0, \dots, s^L) = u_{i_L}(\mathbf{0}) - u_{i_1}(\mathbf{1})$ (if $s^L = \mathbf{0}$) or $\Lambda(s^0, \dots, s^L) = -u_{i_1}(\mathbf{1})$ (if $s^L \neq \mathbf{0}$). Note that if $s^L = \mathbf{0}$, then it must be that $i_L \neq i_1$. Since the choice of a path is arbitrary, except $s^0 = \mathbf{1}$ and $s^L \neq \mathbf{1}$, the conclusion follows from Proposition 1.

It follows that if one of the strict equilibria is an LP-max, then it is strictly preferred to the other strict equilibrium by all but, possibly, one player. It is now straightforward to check that $\mathbf{1} = (1, 1, 1)$ is the LP-max in the three-player unanimity game below.

	0	1
0	6, 2, 2	0, 0, 0
1	0, 0, 0	0, 0, 0

0

	0	1
0	0, 0, 0	0, 0, 0
1	0, 0, 0	3, 8, 8

1 □

In addition to providing insights on which game admits an LP-max, Proposition 1 implies that an LP-max is unique when it exists.

[9] Our characterization of LP-max is closely related to a characterization of feasible solutions to a system of inequalities studied in network flow theory: $x_i - x_j \leqslant w_{ij}$ for $i, j \in \{1, \dots, k\}$ where $\{x_1, \dots, x_k\}$ is a set of variables and w_{ij}'s are constants each specific to the ordered pair (i, j). See, for instance, [20].

Corollary 1. *Every game has at most one LP-max.*

Proof. Consider a path of unilateral deviations starting at s and ending at $s' \neq s$ such that along this path only those players with $s'_i \neq s_i$ change actions and they do so only once, and hence, move from s_i to s'_i. This path is monotonic irrespective of the total order chosen on each S_i since each player changes his action at most once. The reverse of this path is also a monotonic path of unilateral deviations. Since the value of Λ for the reverse of a path is the negative of the value of Λ for the original path, Proposition 1 implies that it is not possible for both s and s' to be LP-max. □

4. Stochastic stability of local potential maximizer

We now state our main result followed by two examples illustrating the theorem.

Theorem 2. *Suppose that a strategy profile s^* is a local potential maximizer for payoff functions $u = (u_1, \dots, u_I)$ with a local potential function $v : S \to \mathbb{R}$. If either* (a) *each u_i is supermodular,*[10] *or* (b) *v is supermodular, then s^* is stochastically stable.*

Example 1. (Continued.) It is clear that the payoff functions of a unanimity game are supermodular with respect to the orderings $0 \leqslant_i 1$ for all $i = 1, \dots, I$. By Theorem 2, strategy profile $s = \mathbf{1}$ (resp. $s = \mathbf{0}$) is the unique stochastically stable state if $u_i(\mathbf{1}) > u_j(\mathbf{0})$ (resp. $u_i(\mathbf{0}) > u_j(\mathbf{1})$) for all i and $j \neq i$. In the $2 \times 2 \times 2$ game of Example 1, $s = \mathbf{1}$ (with payoffs $(3, 8, 8)$) is the unique stochastically stable state. □

Example 2. (See [25].)

	0	1	2
0	6, 6	0, 5	0, 0
1	5, 0	7, 7	5, 5
2	0, 0	5, 5	8, 8

Game $u = (u_1, u_2)$

	0	1	2
0	6	5	0
1	5	7	5
2	0	5	8

Local potential v

It is easy to verify that $(2, 2)$ is the LP-max under the orderings $0 \leqslant_i 1 \leqslant_i 2$, $i = 1, 2$, with a local potential function v exhibited on the right matrix[11] and, in addition, v is supermodular. So $(2, 2)$ is the unique stochastically stable state of the log-linear dynamic. In contrast, $(1, 1)$ is stochastically stable in a version of perturbed best-response (adaptive learning) dynamic studied in [25]. □

As mentioned in the introduction, equilibrium selection for, and robustness of, an LP-max have been established under monotonicity conditions similar to supermodularity in our main

[10] To be clear, the supermodularity here is with respect to the total orders on S_i which makes s^* an LP-max. So, for example, v is supermodular if, for all i, $v(s'_i, s_{-i}) - v(s_i, s_{-i}) \leqslant v(s'_i, s'_{-i}) - v(s_i, s'_{-i})$ whenever $s_i \leqslant_i s'_i$ and $s_{-i} \leqslant_{-i} s'_{-i}$ (i.e., $s_j \leqslant_j s'_j$ for all $j \neq i$). For u_i, the corresponding inequalities should hold for the particular i.

[11] This function was shown in [18] to be a monotone potential function.

result.[12] But whether these results hold without such conditions is not known. Below we demonstrate via an example that an LP-max, in a non-potential game, is not necessarily stochastically stable in the absence of supermodularity. The example also sheds some light on a connection, or a discrepancy, between the notion of LP-max employed in the paper and that of stochastic potential in the tree construction used to determine stochastically stable states.

Example 3. Consider the following 2×3 game.

	0	1	2
0	$1, 1$	$0, -1$	$1, 0.5$
1	$-1, 0$	$1, 1$	$1, 0$

Using Proposition 1, it is easy to check that $(0, 0)$ is an LP-max under the orderings $0 \leqslant_1 1$ and $0 \leqslant_2 1 \leqslant_2 2$. (Hence it is the unique LP-max by Corollary 1.)

Under these orderings, it is easy to check that neither player's payoff function is supermodular. In addition, no local potential function v for $(0, 0)$ can be supermodular. To see this, assume the contrary. Then from Definition 2 (2)-(b-2) applied to u_1 with $s_2 = 0$, we have

$$v(1, 0) - v(0, 0) \geqslant -2. \tag{4.1}$$

Applying Definition 2 (2)-(b-2) now to u_2, we have

$$v(1, 2) - v(1, 1) \geqslant v(0, 2) - v(0, 1) \geqslant 1.5, \tag{4.2a}$$

$$v(1, 1) - v(1, 0) \geqslant 1. \tag{4.2b}$$

The first inequality in (4.2a) follows from the supermodularity assumption on v. Adding (4.2a) and (4.2b) yields $v(1, 2) - v(1, 0) \geqslant 2.5$ which, when added to (4.1), gives $v(1, 2) - v(0, 0) \geqslant 0.5$. This violates Definition 2 (2)-(a).

On the other hand, $(1, 1)$ is the unique stochastically stable state.[13] The proof of the latter fact relies on an appropriate definition of the stochastic potential developed in [1]. For each state $s \in S$, a *revision s-tree* is a tree with S as the set of nodes such that from any other state in S there is a unique path of unilateral deviations terminating at s. Thus a revision s-tree is a tree rooted at s and an edge (s', s'') belongs to it only if $s'' = (s''_i, s'_{-i})$ for some i and $s''_i \neq s'_i$. The set of all revision s-trees is denoted by $\mathcal{T}(s)$. Define the *waste* associated with an edge (s', s'') in a revision s-tree T by[14]

$$\mathrm{w}(s', s'') = \max_{\hat{s}_i \in S_i} u_i(\hat{s}_i, s'_{-i}) - u_i(s''_i, s'_{-i})$$

where i is the player who unilaterally deviates from s' to s''. The waste of a revision s-tree T is defined as the sum of the wastes associated with all edges in T,

$$\mathrm{w}(T) = \sum_{(s', s'') \in T} \mathrm{w}(s', s''). \tag{4.3}$$

[12] See Section 8 for a more detailed description of this point.

[13] In order to show that other equilibrium selection results on LP-max would fail without appropriate monotonicity conditions, some other example must be used. Indeed, $(0, 0)$ is a strict $(1/3, 1/3)$-dominant equilibrium (see Section 5), and hence, is robust to incomplete information (see [7]) and selected by perfect foresight dynamics (see [18]).

[14] The waste corresponds to the *resistance* in the language of perturbed Markov chains. In the adaptive learning model of [25] (resp. [9]) the number of mistakes (resp. mutations), i.e. non-best-response choices of actions, needed to move from one state to another is taken as the resistance.

The *stochastic potential* of a state s is then defined as

$$\pi(s) = \min_{T \in \mathcal{T}(s)} \mathrm{w}(T). \tag{4.4}$$

The main result in [1] asserts that the set of stochastically stable states for the log-linear dynamic is precisely those with the smallest stochastic potential.

Back to the 2×3 game in question, note that there are only two recurrent classes of the best-response dynamic namely, $\{(0,0)\}$ and $\{(1,1)\}$. Consider the $(1,1)$-tree composed of the following three paths of unilateral deviations: (i) $(0,0) \to (0,2) \to (1,2) \to (1,1)$, (ii) $(0,1) \to (1,1)$, and (iii) $(1,0) \to (1,1)$. The waste of this tree is 0.5. On the other hand, any revision $(0,0)$-tree has a waste of at least 1 since any edge $((1,1), s)$ has an associated waste of 1. This shows that $(1,1)$ has the smallest stochastic potential and so is the unique stochastically stable state.

Note that the least costly path of unilateral deviations from $(0,0)$ to $(1,1)$, the path (i) above, is not monotonic. This path contributes to stochastic potential of $(1,1)$ while, in deciding whether $(1,1)$ is an LP-max or not using Proposition 1, such path is not taken into account. □

Example 3 exhibits a non-potential game with an LP-max but without supermodular payoff functions or a supermodular local potential function. As such it belongs outside the class of games to which Theorem 2 is applicable. The next example, taken from [23], exhibits a potential game that is outside applicability of Theorem 2 even though a potential maximizer, a fortiori an LP-max, is stochastically stable. The point is that a potential maximizer is an LP-max with the potential function as one possible local potential function, and there may be others that serve as local potential functions. If one of them is guaranteed to be supermodular, then Theorem 2 would apply to all potential games. The example demonstrates that there is no such guarantee.

Example 4. (See [23].)

	0	1	2
0	4, 4	2, 2	0, 0
1	0, 0	3, 3	0, 0
2	2, 2	0, 0	3, 3

Game $u = (u_1, u_2)$

	0	1	2
0	4	2	0
1	0	3	0
2	2	0	3

Potential function $\hat{v}$

This is a potential game with $s^* = (0,0)$ being the potential maximizer and hence the LP-max. The potential function is exhibited in the matrix on the right. By Theorem 1, $(0,0)$ is stochastically stable. We show below that, first, neither the payoff function nor the potential function is supermodular under any orderings of the strategies, and, second, no local potential function for $(0,0)$ different from $\hat{v}$, if any, can be supermodular.[15]

[15] This game does not have a weighted LP-max with a weighted LP-function that is supermodular, and, in addition, exhibits diminishing marginal returns. This follows from an argument independently developed in [24] and essentially identical to ours in this example (Example 5.17 in [24]). Uno examines a three-player, three-action game which is reduced to our example when one of player 3's action is fixed. He shows that the $3 \times 3 \times 3$ game does not have a weighted LP-max with the two monotonicity properties using inequalities relevant only to players 1 and 2. Thus Uno's argument subsumes that for our example. Nonexistence of a weighted LP-max with supermodularity alone remains to be seen.

Hence Theorem 2 is silent on the stability of $(0,0)$.

To show that u_1 is not supermodular under any orderings, suppose that $0 \leqslant_1 1$. By comparing $u_1(1,s_2) - u_1(0,s_2)$ for $s_2 = 0,1,2$, we see that u_1 is supermodular only if $0 \leqslant_2 2 \leqslant_2 1$. However, $u_1(2,s_2) - u_1(0,s_2)$ is neither increasing nor decreasing in s_2 under the ordering $0 \leqslant_2 2 \leqslant_2 1$. So $1 \leqslant_1 0$, but this is also incompatible with supermodularity of u_1. A similar line of reasoning applies to u_2, and also to $\hat{v}$ as $\hat{v}(\cdot) = u_i(\cdot)$, $i = 1,2$.

Next, suppose that v is a local potential function for $(0,0)$ and it is supermodular with respect to some orderings of strategies, $\leqslant_1, \leqslant_2$.[16] Suppose that $0 \leqslant_1 1$. By Definition 2 (2)-(b-2) applied to u_1, we have

$$v(1,0) - v(0,0) \geqslant -4, \tag{4.5a}$$
$$v(1,1) - v(0,1) \geqslant 1, \tag{4.5b}$$
$$v(1,2) - v(0,2) \geqslant 0. \tag{4.5c}$$

If $1 \leqslant_2 0$, then supermodularity of v and (4.5b) imply $v(1,0) - v(0,0) \geqslant 1$. This violates Definition 2 (2)-(a), i.e., $v(0,0) > v(s)$, $\forall s \neq (0,0)$. So we must have $0 \leqslant_2 1$. Similarly, we must have $0 \leqslant_2 2$. Hence from Definition 2 (2)-(b-2) applied to u_2 we have

$$v(0,1) - v(0,0) \geqslant -2, \tag{4.6a}$$
$$v(1,1) - v(1,0) \geqslant 3, \tag{4.6b}$$
$$v(2,1) - v(2,0) \geqslant -2, \tag{4.6c}$$

and

$$v(0,2) - v(0,0) \geqslant -4, \tag{4.7a}$$
$$v(1,2) - v(1,0) \geqslant 0, \tag{4.7b}$$
$$v(2,2) - v(2,0) \geqslant 1. \tag{4.7c}$$

If $2 \leqslant_1 0$, then supermodularity of v and (4.7c) imply $v(0,2) - v(0,0) \geqslant 1$ which violates Definition 2 (2)-(a). So we must have $0 \leqslant_1 2$ and hence from Definition 2 (2)-(b-2) applied to u_1 we have

$$v(2,0) - v(0,0) \geqslant -2, \tag{4.8a}$$
$$v(2,1) - v(0,1) \geqslant -2, \tag{4.8b}$$
$$v(2,2) - v(0,2) \geqslant 3. \tag{4.8c}$$

If $2 \leqslant_2 1$, then supermodularity of v and (4.8c) imply $v(2,1) - v(0,1) \geqslant 3$. Add this inequality to (4.6a) and get $v(2,1) - v(0,0) \geqslant 1$ which violates Definition 2 (2)-(a). So we must have $1 \leqslant_2 2$, hence $0 \leqslant_2 1 \leqslant_2 2$, and from Definition 2 (2)-(b-2) applied to u_2 we have

$$v(0,2) - v(0,1) \geqslant -2, \tag{4.9a}$$
$$v(1,2) - v(1,1) \geqslant -3, \tag{4.9b}$$
$$v(2,2) - v(2,1) \geqslant 3. \tag{4.9c}$$

[16] It is not difficult to check that, even if one uses the set-valued formulation of LP-max (Definition 4 in Appendix B), $(0,0)$ can be an LP-max only under the ordered domain with the finest partitions of the strategy sets.

If $2 \leqslant_1 1$, then supermodularity of v and (4.9c) imply $v(1,2) - v(1,1) \geqslant 3$. Add this inequality to (4.5b) to get $v(1,2) - v(0,1) \geqslant 4$ which, by adding (4.6a) gives $v(1,2) - v(0,0) \geqslant 2$, violating Definition 2 (2)-(a). So we must have $1 \leqslant_1 2$, hence $0 \leqslant_1 1 \leqslant_1 2$, and from Definition 2 (2)-(b-2) applied to u_1 we have

$$v(2,0) - v(1,0) \geqslant 2, \tag{4.10a}$$

$$v(2,1) - v(1,1) \geqslant -3, \tag{4.10b}$$

$$v(2,2) - v(1,2) \geqslant 3. \tag{4.10c}$$

Now, supermodularity of v, $0 \leqslant_2 1$, and (4.10a) imply $v(2,1) - v(1,1) \geqslant 2$. Add this to (4.6b) to get $v(2,1) - v(1,0) \geqslant 5$ which, by adding (4.5a), gives $v(2,1) - v(0,0) \geqslant 1$ which violates Definition 2 (2)-(a). By a symmetric argument it can be shown that assuming $1 \leqslant_1 0$ implies $2 \leqslant_i 1 \leqslant_i 0$, $i = 1, 2$, and leads to a violation of Definition 2 (2)-(a). □

Examples 3 and 4 together demonstrate the scope of our theorem. More importantly, they highlight the divergence between the potential maximizer and the LP-max that makes supermodularity (resp. related monotonicity conditions) necessary for stochastic stability (resp. equilibrium selection and robustness).

We will present the proof of Theorem 2 in Section 6. Before closing this section we give a brief explanation of the key ideas to help the reader understand the formal proof better and for those who wish to skip the technical details. The basic principle is most simply illustrated in the case where a local potential maximizer $s^* = (s_1^*, \ldots, s_I^*)$ is such that each s_i^* is the maximum element of the associated ordering $\leqslant_i$. Then, whenever $s_i \leqslant_i s_i'$, the difference in payoff to player i caused by switching unilaterally from s_i to s_i' is bounded from below by the corresponding difference in the values of the local potential function v (Definition 2 (2)-(b-1)).

Now consider two log-linear dynamics, one for the original payoff functions, call it the u-dynamic, and the other, the v-dynamic, for the local potential function v as the payoff function common to all players. In a log-linear dynamic, the difference in payoffs from unilaterally switching from s_i to s_i' is (proportional to) the log ratio of the choice probabilities between the two actions (Eq. (2.1)). So, the ratio of the probability of choosing s_i' over the probability of choosing s_i in the u-dynamic is at least as large as the corresponding ratio in the v-dynamic. As a consequence, after the first transition starting at the same initial state, the u-dynamic is more likely to be in a higher state (in $\leqslant_i$ orderings) than the v-dynamic is.

At this point, the two dynamics will have different distributions of states. So, we cannot conclude that, after the second transition, the u-dynamic is more likely to be in a higher state than the v-process is. Supermodularity assumption is precisely what is needed to remedy this situation. For instance, if u_i's are supermodular, then the u-dynamic is more likely to move toward a higher state than a lower state when it starts from a distribution of states under which higher states are more likely. Hence, the u-dynamic is also more likely to move toward a higher state than the v-dynamic because the u-dynamic is more likely to be in a higher state already. A similar reasoning applies in case v is supermodular. Since the v-dynamic tends toward s^* (Theorem 1), and this is the highest possible state, the conclusion of our main theorem follows.

In short, we show that, in the presence of a local potential maximizer, (a) two Markov chains (one-step transition matrices), one for u_i's and the other for v, are stochastically ordered (Proposition 3) and (b) the supermodularity assumption preserves this ordering for all t-step transition matrices (Proposition 4). While we use only elementary tools of finite Markov chains, interested readers can find additional information on comparison of Markov processes, general stochastic processes and dynamical systems in [8,11,15,21,22].

5. Local potential maximizer with non-constant weights

In this section we discuss stochastic stability, and lack thereof, of a weighted potential maximizer (see [12]) and its generalization, a weighted local potential maximizer. The discussion serves to better understand the nature of the log-linear dynamic, especially its sensitivity to the cardinality of payoffs.

A strategy profile s^* is a weighted potential maximizer if there exists a function (potential function) $v : S \to \mathbb{R}$ and weights $w_i \geqslant 0$, $i = 1, \dots, I$, with the property that s^* maximizes v and

$$u_i\left(s_i', s_{-i}\right) - u_i(s_i, s_{-i}) = w_i\left(v\left(s_i', s_{-i}\right) - v(s_i, s_{-i})\right) \tag{5.1}$$

for all $i = 1, \dots, I$, $s_i, s_i' \in S_i$ and $s_{-i} \in S_{-i}$.

Definition 3. (See [13].) A strategy profile $s^* = (s_1^*, \dots, s_I^*)$ is a *weighted local potential maximizer* (weighted LP-max)[17] for payoff functions $u = (u_1, \dots, u_I)$ if there exist

(1) a total order $\leqslant_i$ on each S_i, $i = 1, \dots, I$,
(2) a function $v : S \to \mathbb{R}$, and a collection of nonnegative numbers $\{w_i(s_i) \mid s_i \in S_i\}$ (weights) for each $i = 1, \dots, I$ such that
 (a) $v(s^*) > v(s)$ for all $s \neq s^*$,
 (b) for each i, and every $s_{-i} \in S_{-i}$,
  (b-1) if $s_i \leqslant_i s_i^*$ and $s_i \neq s_i^*$, then $w_i(s_i)(v(s_i^+, s_{-i}) - v(s_i, s_{-i})) \leqslant u_i(s_i^+, s_{-i}) - u_i(s_i, s_{-i})$,
  (b-2) if $s_i^* \leqslant_i s_i$ and $s_i \neq s_i^*$, then $w_i(s_i)(v(s_i, s_{-i}) - v(s_i^-, s_{-i})) \geqslant u_i(s_i, s_{-i}) - u_i(s_i^-, s_{-i})$,
 where s_i^+ (resp. s_i^-) is the smallest (resp. largest) element of S_i that is larger (resp. smaller) than s_i.

If s^* is a weighted potential maximizer with a potential function v and weights w_i, $i = 1, \dots, I$, then it is a weighted local potential maximizer under any orderings of strategies with v as a local potential function and weights $w_i(s_i) = w_i$ for all i and s_i. The first example, due to [1], demonstrates that a weighted potential maximizer may not be stochastically stable and hence Theorem 1 does not extend to weighted potential games. Further, the associated potential function in this example can be made supermodular under some orderings of strategies and so some games with weighted LP-max lies outside the scope of Theorem 2.

Example 5. (See [1].)

	0	1
0	2, 2	0, 0
1	0, 0	10, 1

$u = (u_1, u_2)$

	0	1
0	2	−6
1	0	4

v

[17] In [13], this is taken as the definition of an LP-max. In order to avoid confusion with Definition 2, we use the modifier *weighted*.

Strategy profile $(1,1)$ is the weighted potential maximizer with the potential function v shown on the right matrix and the weights $w_1 = 1$ and $w_2 = \frac{1}{4}$. With respect to orderings $0 \leqslant_i 1$, $i = 1, 2$, the function v is supermodular.

It is easy to compute stochastic potentials of $(0,0)$ and $(1,1)$ using the method discussed in Example 3: $\pi((0,0)) = 1$ and $\pi((1,1)) = 2$. Therefore $(0,0)$ is the unique stochastically stable state.

Could the stochastic stability of $(0,0)$ be demonstrated as a consequence of Theorem 2? The answer is no. In fact, there is no LP-max (according to Definition 2) for this game. For example, take the orderings $0 \leqslant_i 1$, $i = 1, 2$. For a path $(s^0, s^1, s^2) = ((1,1), (1,0), (0,0))$, we have $\Lambda((s^0, s^1, s^2)) = (0-1) + (2-0) = 1 > 0$. So Proposition 1 implies that $(1,1)$ is not an LP-max. Similar lines of reasoning work for other orderings and also for showing that $(0,0)$ is not an LP-max. □

It is easy to verify via a simple modification to the proof of Theorem 1 (see [4]) that if we allow the parameter β to vary across players, $\beta_1 = \beta$ and $\beta_2 = 4\beta$ for Example 5, then the weighted potential maximizer becomes stochastically stable, i.e., $\mu_{(1,1)}(u, (\beta, 4\beta)) \to 1$ as $\beta \to \infty$. More generally, consider a log-linear dynamic where the parameter β for player i's stochastic choice rule (2.1) (resp. (2.2)) is replaced by β_i. Again the Markov chain generated by this dynamic is irreducible and aperiodic. The proof of the next proposition is virtually identical to that of Theorem 2 and thus omitted.

Proposition 2. *Suppose that s^* is a weighted LP-max for $u = (u_1, \ldots, u_I)$ with a local potential function v and weights $w_i(s_i) \equiv w_i$ for each $i = 1, \ldots, I$ and $s_i \in S_i$. If u or v is supermodular, then s^* is the unique stochastically stable state of the log-linear dynamic with coefficients $\beta_i = \frac{\beta}{w_i}$, $i = 1, \ldots, I$.*

A complementary way of looking at Proposition 2 is that we can make the strategy profile $(1,1)$ stochastically stable in the original dynamic with a common β by applying an affine transformation to player 1's payoffs. The lack of invariance of the stochastically stable state to affine transformations is a distinctive feature of the log-linear dynamic.

To illustrate this point further, consider a 2-by-2 game with $S_1 = S_2 = \{0, 1\}$ with $(0,0)$ and $(1,1)$ being strict equilibria, just as in Example 5. Now modify the log-linear choice rule, (2.1), so that the log likelihood ratio of choosing one action over another is linearly proportional to the *relative* difference of payoffs they yield.[18]

$$\ln \frac{p_i^r(1 \mid s : u_i, \beta)}{p_i^r(0 \mid s : u_i, \beta)} = \beta \left(\frac{u_i(1, s_j) - u_i(0, s_j)}{D_i} \right), \quad i = 1, 2, \ j \neq i, \tag{5.2}$$

where $D_i > 0$ is the sum of the payoff losses caused by unilaterally deviating from each equilibrium: $D_1 = [u_1(1,1) - u_1(0,1)] + [u_1(0,0) - u_1(1,0)]$ and $D_2 = [u_2(1,1) - u_2(1,0)] + [u_2(0,0) - u_2(0,1)]$. The Markov chain on S derived from this choice rule, a *relative log-linear dynamic*, is irreducible and aperiodic with a unique invariant distribution $\mu^r(u, \beta)$.

[18] We could study this choice rule and the resulting dynamic for general normal form games, but it is beyond the scope of this paper.

It is clear that (5.2) is unchanged if we apply an affine transformation to the payoff function u_i. Note also that

$$\ln \frac{p_i^r(1 \mid s : u_i, \beta)}{p_i^r(0 \mid s : u_i, \beta)} = \ln \frac{p_i(1 \mid s : u_i^r, \beta)}{p_i(0 \mid s : u_i^r, \beta)}$$

where $u_i^r(s) = \frac{u_i(s)}{D_i}$, and so $\mu^r(u, \beta) = \mu(u^r, \beta)$. That is, the relative log-linear dynamic with the original payoff functions $u = (u_1, u_2)$ is the original log-linear dynamic with the relative payoff functions $u^r = (u_1^r, u_2^r)$.

If $(0, 0)$ (resp. $(1, 1)$) is a weighted potential maximizer for $u = (u_1, u_2)$, then it is a (exact) potential maximizer for $u^r = (u_1^r, u_2^r)$ and so Theorem 1 implies that $(0, 0)$ (resp. $(1, 1)$) is the stochastically stable state of the relative log-linear dynamic. More generally, if $(0, 0)$ (resp. $(1, 1)$) is a weighted local potential maximizer for $u = (u_1, u_2)$, then $(0, 0)$ (resp. $(1, 1)$) is the stochastically stable state of the relative log-linear dynamic. The proof is a straightforward application of Theorem 2 (see [17]).[19]

Let us make a brief remark on a notion of equilibrium closely related to LP-max: **p**-dominant equilibrium. Let $\mathbf{p} = (p_1, \ldots, p_I)$ with $0 \leqslant p_i < 1$, $i = 1, \ldots, I$. A strategy profile $s^* \in S$ is a **p**-dominant (resp. strict **p**-dominant) equilibrium if, for every i, s_i^* is a (resp. the unique) best response to any mixture over S_{-i} that puts probability at least (resp. strictly greater than) p_i on s_{-i}^*.

For games where each player has two actions, if s^* is a strict **p**-dominant equilibrium with $p_1 + \cdots + p_I < 1$, then s^* is a weighted LP-max [14] with a supermodular local potential function. Moreover, for 2-by-2 games, a strict (p_1, p_2)-dominant equilibrium with $p_1 + p_2 < 1$ is a weighted LP-max with strictly positive weights [14] and so it is the stochastically stable state of the relative log-linear dynamic.

Links between **p**-dominant equilibrium and the original form of log-linear dynamic is more tenuous. For instance, a (p_1, p_2)-dominant equilibrium in a 2-by-2 game with $\max\{p_1, p_2\} < \frac{1}{2}$ is stochastically stable in the original dynamic. In general, however, a **p**-dominant equilibrium with $p_1 + \cdots + p_I < 1$ is not necessarily stochastically stable, even for symmetric games with $\max_i p_i < 1/2$. See [17] for the details.

6. Proof of the main theorem

We carry out the proof of Theorem 2 in a series of steps that examine the relationship between two Markov chains, $Q(v, \beta)$ for the game with a local potential v as a common payoff function (v-dynamic), and $Q(u, \beta)$ for the game with given payoff functions $u = (u_1, \ldots, u_I)$ (u-dynamic). By Theorem 1, the stochastically stable state of the v-dynamic is s^*, the local potential maximizer. In order to show that s^* is also stochastically stable in the u-dynamic, we establish a stochastic ordering between the two dynamics, not only in the limit as $\beta \to \infty$, but for each $\beta < \infty$.

6.1. Stochastic orders

Let $\leqslant_i$ be a total order on S_i, $i = 1, \ldots, I$. The associated product order on $S \equiv S_1 \times \cdots \times S_I$ is denoted by $\leqslant_S$. Thus $(s_1, \ldots, s_I) \leqslant_S (s_1', \ldots, s_I')$ iff $s_i \leqslant_i s_i'$ for all i. We say that a set $T_i \subset S_i$

[19] Supermodularity condition is trivially satisfied for games under consideration.

(resp. $T \subset S$) is increasing if $s_i \in T_i$ and $s_i \leqslant_i s_i'$ imply $s_i' \in T_i$ (resp. $s \in T$ and $s \leqslant_S s'$ imply $s' \in T$). Clearly, $T_i \subset S_i$ is increasing if, and only if, $T_i = \{s_i \in S_i \mid s_i' \leqslant_i s_i\}$ for some $s_i' \in S_i$, and $T \subset S$ is increasing if, and only if, it is a union of sets of the form $T_1 \times \cdots \times T_I$ where each $T_i \subset S_i$ is increasing. In particular, if $T \subset S$ is increasing then $T_i = \{s_i \in S_i \mid (s_i, s_{-i}) \in T \text{ for some } s_{-i}\}$ is increasing for every i.

Let $\Delta(S_i)$ (resp. $\Delta(S)$) be the set of all probability distributions on S_i (resp. S). For any $\mu_i, \nu_i \in \Delta(S_i)$, we write $\mu_i \preccurlyeq_i \nu_i$ if $\mu_i(T_i) \leqslant \nu_i(T_i)$ for any increasing set $T_i \subset S_i$.[20] Similarly, for any $\mu, \nu \in \Delta(S)$, we write $\mu \preccurlyeq \nu$ if $\mu(T) \leqslant \nu(T)$ for any increasing set $T \subset S$. The next lemma is standard and stated without proof.

Lemma 1. *For any $\mu_i, \nu_i \in \Delta(S_i)$, $\mu_i \preccurlyeq_i \nu_i$ if, and only if,*

$$\sum_{s_i \in S_i} \mu_i(s_i)\phi(s_i) \leqslant \sum_{s_i \in S_i} \nu_i(s_i)\phi(s_i)$$

for every nondecreasing function $\phi : S_i \to \mathbb{R}$. Similarly, for any $\mu, \nu \in \Delta(S)$, $\mu \preccurlyeq \nu$ if, and only if, $\sum_{s \in S} \mu_s \psi(s) \leqslant \sum_{s \in S} \nu_s \psi(s)$ for any nondecreasing function $\psi : S \to \mathbb{R}$.

Let P and Q be transition matrices of Markov chains on S. Denote the "s-th row" of P and Q by $p(\cdot \mid s)$ and $q(\cdot \mid s)$, respectively. We write $P \preccurlyeq Q$ if $p(\cdot \mid s) \preccurlyeq q(\cdot \mid s)$ for every $s \in S$ (every row of Q first-order stochastically dominates the corresponding row of P).

Lemma 2. *If $P \preccurlyeq Q$, then $\mu P \preccurlyeq \mu Q$ for any $\mu \in \Delta(S)$.*

Proof. Let $T \subset S$ be an increasing set and $\mu = (\mu_s)_{s \in S} \in \Delta(S)$. Then

$$(\mu P)(T) = \sum_{s \in S} \mu_s p(T \mid s) \leqslant \sum_{s \in S} \mu_s q(T \mid s) = (\mu Q)(T)$$

where the inequality follows from the assumption $P \preccurlyeq Q$. □

6.2. Comparison of Markov chains $Q(u, \beta)$ and $Q(v, \beta)$

Now suppose that $s^* = (s_1^*, \dots, s_I^*)$ is a local potential maximizer for payoff functions $u = (u_1, \dots, u_I)$ with the associated total orders $\leqslant_i$ on S_i, $i = 1, \dots, I$, and a local potential function $v : S \to \mathbb{R}$. The main idea of the proof is as follows. We split the Markov chain for the v-dynamic into two chains, $Q^-(v, \beta)$ and $Q^+(v, \beta)$. In the dynamic associated with $Q^-(v, \beta)$ (resp. $Q^+(v, \beta)$), player i (if he is given a chance to revise his action) can switch to s_i only if $s_i \leqslant_i s_i^*$ (resp. $s_i^* \leqslant_i s_i$). The Markov chain for the u-dynamic, $Q(u, \beta)$, is shown to be "sandwiched" between $Q^-(v, \beta)$ and $Q^+(v, \beta)$ in the stochastic order $\preccurlyeq$ as defined above. Under the additional assumption of supermodularity (of v or each u_i) this order is preserved for the corresponding t-step transition matrices. Letting t tend to infinity the invariant distribution for $Q(u, \beta)$ is shown to be similarly sandwiched between the invariant distributions for $Q^-(v, \beta)$ and $Q^+(v, \beta)$ for every $\beta > 0$. Finally, both the invariant distributions of $Q^-(v, \beta)$ and $Q^+(v, \beta)$ are shown to converge to a point mass on s^* as $\beta \to \infty$ and the proof is complete.

Set $S_i^- = \{s_i \in S_i \mid s_i \leqslant_i s_i^*\}$ and $S^- = S_1^- \times \cdots \times S_I^-$. Define a modified log-linear choice rule $p_i^-(s_i' \mid s : f_i, \beta)$, where $f_i = u_i$ or v, by

[20] This is just the definition of first order stochastic dominance.

$$p_i^-\big(s_i' \mid s : f_i, \beta\big) = \mathbf{I}\big(s_i' \in S_i^-\big) \frac{p_i(s_i' \mid s : f_i, \beta)}{p_i(S_i^- \mid s : f_i, \beta)} \tag{6.1}$$

or, more specifically,

$$p_i^-\big(s_i' \mid s : f_i, \beta\big) = \begin{cases} \dfrac{e^{\beta f_i(s_i', s_{-i})}}{\sum_{s_i'' \in S_i^-} e^{\beta f_i(s_i'', s_{-i})}} & \text{if } s_i' \in S_i^-, \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

Let $Q^-(f, \beta) = (q_{ss'}^-(f, \beta))_{s,s' \in S}$ where $q_{ss'}^-(f, \beta)$ is computed by substituting (6.1) into (2.3). Thus $Q^-(f, \beta)$ is the transition matrix for a log-linear dynamic where transitions to states outside of S^- are prohibited.

Similarly, let $S_i^+ = \{s_i \in S_i \mid s_i^* \leqslant_i s_i\}$, $S^+ = S_1^+ \times \cdots \times S_I^+$ and define

$$p_i^+\big(s_i' \mid s : f_i, \beta\big) = \mathbf{I}\big(s_i' \in S_i^+\big) \frac{p_i(s_i' \mid s : f_i, \beta)}{p_i(S_i^+ \mid s : f_i, \beta)} = \begin{cases} \dfrac{e^{\beta f_i(s_i', s_{-i})}}{\sum_{s_i'' \in S_i^+} e^{\beta f_i(s_i'', s_{-i})}} & \text{if } s_i' \in S_i^+, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

The corresponding transition probability is denoted by $q_{ss'}^+(f, \beta)$ and we set $Q^+(f, \beta) = (q_{ss'}^+(f, \beta))_{s,s' \in S}$.

For each $s \in S$, let $q_s(f, \beta) = (q_{ss'}(f, \beta))_{s' \in S}$ be the "s-th row" of $Q(f, \beta)$, and, similarly, let $q_s^-(f, \beta) = (q_{ss'}^-(f, \beta))_{s' \in S}$ and $q_s^+(f, \beta) = (q_{ss'}^+(f, \beta))_{s' \in S}$.

Proposition 3. $Q^-(v, \beta) \preccurlyeq Q^-(u, \beta) \preccurlyeq Q(u, \beta) \preccurlyeq Q^+(u, \beta) \preccurlyeq Q^+(v, \beta)$.

Proof. We will only show $Q^-(v, \beta) \preccurlyeq Q^-(u, \beta) \preccurlyeq Q(u, \beta)$ as the proof of $Q(u, \beta) \preccurlyeq Q^+(u, \beta) \preccurlyeq Q^+(v, \beta)$ is similar.

First, we verify $Q^-(v, \beta) \preccurlyeq Q^-(u, \beta)$, i.e., $q_s^-(v, \beta) \preccurlyeq q_s^-(u, \beta)$ for all $s \in S$. Since $q_{ss'}^-(f, \beta) = \sum_{i=1}^I \rho_i \mathbf{I}(s_{-i}' = s_{-i}) p_i^-(s_i' \mid s : f, \beta)$ where $f = v$ or u_i, it suffices to show that $p_i^-(\cdot \mid s : v, \beta) \preccurlyeq_i p_i^-(\cdot \mid s : u_i, \beta)$ for all i and s. Take $s_i', s_i'' \in S_i$ with $s_i' \leqslant_i s_i''$. If $s_i^* \leqslant_i s_i''$ and $s_i'' \neq s_i^*$, then by the definition of p_i^- we have

$$p_i^-\big(s_i'' \bigm| s : v, \beta\big) p_i^-\big(s_i' \bigm| s : u_i, \beta\big) = 0 = p_i^-\big(s_i' \bigm| s : v, \beta\big) p_i^-\big(s_i'' \bigm| s : u_i, \beta\big).$$

On the other hand, if $s_i'' \leqslant_i s_i^*$, then, by (6.2) and since s^* is an LP-max (Definition 2 (2)-(b-1)), we have

$$\frac{p_i^-(s_i'' \mid s : v, \beta)}{p_i^-(s_i' \mid s : v, \beta)} = \frac{e^{\beta v(s_i'', s_{-i})}}{e^{\beta v(s_i', s_{-i})}} \leqslant \frac{e^{\beta u_i(s_i'', s_{-i})}}{e^{\beta u_i(s_i', s_{-i})}} = \frac{p_i^-(s_i'' \mid s : u_i, \beta)}{p_i^-(s_i' \mid s : u_i, \beta)}.$$

Thus $p_i^-(\cdot \mid s : v, \beta)$ is smaller than $p_i^-(\cdot \mid s : u_i, \beta)$ in the likelihood ratio order. Hence, $p_i^-(\cdot \mid s : v, \beta) \preccurlyeq_i p_i^-(\cdot \mid s : u_i, \beta)$ (the first order stochastic dominance) follows from the standard argument.

We now turn to the proof of $Q^-(u, \beta) \preccurlyeq Q(u, \beta)$, i.e., $q_s^-(u, \beta) \preccurlyeq q_s(u, \beta)$ for all $s \in S$. Again, it is enough to show that $p_i^-(\cdot \mid s : u_i, \beta) \preccurlyeq_i p_i(\cdot \mid s : u_i, \beta)$. Let $T_i \subset S_i$ be an increasing set. From the definition of S_i^-, it is easily seen that either $T_i \cap S_i^- = \varnothing$ or $T_i \cup S_i^- = S_i$. If $T_i \cap S_i^- = \varnothing$, then $p_i^-(T_i \mid s : u_i, \beta) = 0 \leqslant p_i(T_i \mid s : u_i, \beta)$. If $T_i \cup S_i^- = S_i$, then $T_i \setminus S_i^- = S_i \setminus S_i^-$. So (omitting references to s, u_i, and β)

$$\begin{aligned} p_i(T_i)p_i\big(S_i^-\big) &= \big[p_i\big(T_i \cap S_i^-\big) + p_i\big(T_i \setminus S_i^-\big)\big]p_i\big(S_i^-\big) \\ &= \big[p_i\big(T_i \cap S_i^-\big) + \big(1 - p_i\big(S_i^-\big)\big)\big]p_i\big(S_i^-\big) \\ &= \big(1 - p_i\big(S_i^-\big)\big)p_i\big(S_i^-\big) + p_i\big(S_i^-\big)p_i\big(T_i \cap S_i^-\big) \geqslant p_i\big(T_i \cap S_i^-\big) \end{aligned}$$

and hence $p_i^-(T_i) \leqslant p_i(T_i)$. □

Lemma 3. *Let $T \subset S$ be an increasing set and, for each $i = 1, \dots, I$ and $s \in S$, let $T_i(s) = \{s_i' \in S_i \mid (s_i', s_{-i}) \in T\}$. If the local potential function v is supermodular, then the two functions $\phi_i : S \to \mathbb{R}$ and $\phi : S \to \mathbb{R}$ defined below are nondecreasing*:

$$\phi_i(s) = p_i^-\big(T_i(s) \,\big|\, s : v, \beta\big), \qquad \phi(s) = \sum_{s' \in T} q_{ss'}^-(v, \beta).$$

Proof. Since

$$\begin{aligned} \phi(s) = \sum_{s' \in T} q_{ss'}^-(v, \beta) &= \sum_{s' \in T} \sum_{i=1}^{I} \rho_i \mathbf{I}\big(s_{-i}' = s_{-i}\big) p_i^-\big(s_i' \,\big|\, s : v, \beta\big) \\ &= \sum_{i=1}^{I} \rho_i \sum_{s' \in T} \mathbf{I}\big(s_{-i}' = s_{-i}\big) p_i^-\big(s_i' \,\big|\, s : v, \beta\big) \\ &= \sum_{i=1}^{I} \rho_i \sum_{s_i' \in T_i(s)} p_i^-\big(s_i' \,\big|\, s : v, \beta\big) = \sum_{i=1}^{I} \rho_i \phi_i(s), \end{aligned}$$

it is enough to prove the claim only for ϕ_i. If $s \leqslant_S s'$ and $\bar{s}_i \leqslant_i \hat{s}_i$, then by the supermodularity we have $v(\hat{s}_i, s_{-i}) - v(\bar{s}_i, s_{-i}) \leqslant v(\hat{s}_i, s_{-i}') - v(\bar{s}_i, s_{-i}')$ and hence, by (6.2),

$$\frac{p_i^-(\hat{s}_i \mid s : v, \beta)}{p_i^-(\bar{s}_i \mid s : v, \beta)} = \frac{e^{\beta v(\hat{s}_i, s_{-i})}}{e^{\beta v(\bar{s}_i, s_{-i})}} \leqslant \frac{e^{\beta v(\hat{s}_i, s_{-i}')}}{e^{\beta v(\bar{s}_i, s_{-i}')}} = \frac{p_i^-(\hat{s}_i \mid s' : v, \beta)}{p_i^-(\bar{s}_i \mid s' : v, \beta)}.$$

It follows that $p_i^-(\cdot \mid s : v, \beta) \preccurlyeq_i p_i^-(\cdot \mid s' : v, \beta)$. Since T is increasing, $T_i(s)$ is increasing for any $s \in S$. In addition, $T_i(s) \subset T_i(s')$ whenever $s \leqslant_S s'$. Hence

$$\phi_i(s) = p_i^-\big(T_i(s) \,\big|\, s : v, \beta\big) \leqslant p_i^-\big(T_i(s) \,\big|\, s' : v, \beta\big) \leqslant p_i^-\big(T_i\big(s'\big) \,\big|\, s' : v, \beta\big) = \phi_i\big(s'\big)$$

as claimed. □

Corollary 2. *Suppose that v is supermodular. Then for any $\mu, \nu \in \Delta(S)$ with $\mu \preccurlyeq \nu$, we have $\mu Q^-(v, \beta) \preccurlyeq \nu Q^-(v, \beta)$ and $\mu Q^+(v, \beta) \preccurlyeq \nu Q^+(v, \beta)$.*

Proof. Let $p = \mu Q^-(v, \beta)$ and $q = \nu Q^-(v, \beta)$, and let $T \subset S$ be an increasing set. Then

$$p(T) = \sum_{s' \in T} \sum_{s \in S} \mu_s q_{ss'}^-(v, \beta) = \sum_{s \in S} \mu_s \sum_{s' \in T} q_{ss'}^-(v, \beta) = \sum_{s \in S} \mu_s \phi(s)$$

where $\phi(s)$ is defined as in Lemma 3. Similarly, $q(T) = \sum_{s \in S} \nu_s \phi(s)$. Since $\mu \preccurlyeq \nu$ and ϕ is nondecreasing by Lemma 3, we have $p(T) \leqslant q(T)$ by Lemma 1. Hence $p \preccurlyeq q$. The proof for $Q^+(v, \beta)$ is similar and omitted. □

Proposition 4. *Suppose that either v is supermodular or u_i is supermodular for each $i = 1, \dots, I$. Then, for any $\mu, \nu \in \Delta(S)$ with $\mu \preccurlyeq \nu$, we have $\mu Q^-(v, \beta) \preccurlyeq \nu Q^-(u, \beta) \preccurlyeq \nu Q(u, \beta)$ and $\mu Q(u, \beta) \preccurlyeq \mu Q^+(u, \beta) \preccurlyeq \nu Q^+(v, \beta)$.*

Proof. We carry out the proof only for Q^-. Take $\mu, \nu \in \Delta(S)$ with $\mu \preccurlyeq \nu$ and let $T \subset S$ be an increasing set. Suppose first that v is supermodular. Then $\mu Q^-(v, \beta) \preccurlyeq \nu Q^-(v, \beta)$ by Corollary 2, and $\nu Q^-(v, \beta) \preccurlyeq \nu Q^-(u, \beta) \preccurlyeq \nu Q(u, \beta)$ by Proposition 3 and Lemma 2. Hence $\mu Q^-(v, \beta) \preccurlyeq \nu Q^-(u, \beta) \preccurlyeq \nu Q(u, \beta)$.

Next, suppose that each u_i is supermodular. Let $p = \mu Q^-(v, \beta)$, $q = \nu Q^-(u, \beta)$ and let $T \subset S$ be an increasing set. Define a function $\psi : S \to \mathbb{R}$ by $\psi(s) = \sum_{s' \in T} q^-_{ss'}(u, \beta)$. Recall the function ϕ from Lemma 3, $\phi = \sum_{s' \in T} q^-_{ss'}(v, \beta)$. Since $Q^-(v, \beta) \preccurlyeq Q^-(u, \beta)$ (Proposition 3) we have $\phi(s) \leqslant \psi(s)$ for every $s \in S$. So $p(T) = \sum_{s \in S} \mu_s \phi(s) \leqslant \sum_{s \in S} \mu_s \psi(s)$. By an argument similar to that in the proof of Lemma 3 it follows that ψ is nondecreasing. Since $\mu \preccurlyeq \nu$, Lemma 1 implies $\sum_{s \in S} \mu_s \psi(s) \leqslant \sum_{s \in S} \nu_s \psi(s) = q(T)$. So $p(T) \leqslant q(T)$ and hence $\mu Q^-(v, \beta) \preccurlyeq \nu Q^-(u, \beta)$. The proof is complete by noting that $\nu Q^-(u, \beta) \preccurlyeq \nu Q(u, \beta)$ by Proposition 3 and Lemma 2. $\square$

The next lemma is a straightforward variation of a result by [4].

Lemma 4. (a) *$Q^-(v, \beta)$ has a unique invariant distribution given by*

$$\mu^-_s(v, \beta) = \begin{cases} \frac{e^{\beta v(s)}}{\sum_{s' \in S^-} e^{\beta v(s')}} & \text{if } s \in S^-, \\ 0 & \text{otherwise} \end{cases} \tag{6.4}$$

and, similarly, $Q^+(v, \beta)$ has a unique invariant distribution given by

$$\mu^+_s(v, \beta) = \begin{cases} \frac{e^{\beta v(s)}}{\sum_{s' \in S^+} e^{\beta v(s')}} & \text{if } s \in S^+, \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

(b) *The unique stochastically stable state of $Q^-(v, \beta)$ and $Q^+(v, \beta)$ is s^*, i.e.,*

$$\lim_{\beta \to \infty} \mu^-_{s^*}(v, \beta) = \lim_{\beta \to \infty} \mu^+_{s^*}(v, \beta) = 1. \tag{6.6}$$

Proof. Since (b) is a straightforward consequence of (6.4), (6.5) and $v(s^*) > v(s)$ for every $s \neq s^*$ (Definition 2), we will only show (a). Again, we carry out the proof only for $Q^-(v, \beta)$.

Observe first that $Q^-(v, \beta)$ has a unique recurrent class S^- and it is aperiodic. Hence, $Q^-(v, \beta)$ has a unique invariant distribution whose support is S^-. To see that $\mu^-(v, \beta)$ defined by (6.4) is indeed an invariant distribution, hence the unique one, it is enough to check that it satisfies the detailed balance condition, $\mu^-_s(v, \beta) q^-_{ss'}(v, \beta) = \mu^-_{s'}(v, \beta) q^-_{s's}(v, \beta)$ for any $s, s' \in S$.

If either $s \notin S^-$ or $s' \notin S^-$, or, if s and s' differ in more than one coordinate, then the both sides of the equality are 0. If $s, s' \in S^-$ and $s' = (s'_i, s_{-i})$ (so $s = (s_i, s'_{-i})$), then

$$\begin{aligned} \mu^-_s(v, \beta) q^-_{ss'}(v, \beta) &= \frac{e^{\beta v(s)}}{\sum_{s'' \in S^-} e^{\beta v(s'')}} \rho_i \frac{e^{\beta v(s'_i, s_{-i})}}{\sum_{s''_i \in S^-_i} e^{\beta v(s''_i, s_{-i})}} \\ &= \frac{e^{\beta v(s')}}{\sum_{s'' \in S^-} e^{\beta v(s'')}} \rho_i \frac{e^{\beta v(s_i, s'_{-i})}}{\sum_{s''_i \in S^-_i} e^{\beta v(s''_i, s'_{-i})}} = \mu^-_{s'}(v, \beta) q^-_{s's}(v, \beta). \qquad \square \end{aligned}$$

We are now ready to complete the proof of Theorem 2. Recall that $Q(u,\beta)$ is irreducible and aperiodic, and its unique invariant distribution is denoted by $\mu(u,\beta)$.

Proof of Theorem 2. We will show that $\mu^-(v,\beta) \preccurlyeq \mu(u,\beta) \preccurlyeq \mu^+(v,\beta)$ which, together with Lemma 4 (b), implies that $\lim_{\beta\to\infty} \mu_{s^*}(u,\beta) = 1$.

By Proposition 3 we have $Q^-(v,\beta) \preccurlyeq Q(u,\beta)$. Suppose that v is supermodular or each u_i is supermodular. Let $\mu, \nu \in \Delta(S)$ be such that $\mu \preccurlyeq \nu$. Then $\mu Q^-(v,\beta) \preccurlyeq \nu Q(u,\beta)$ by Proposition 4. Repeated application of Proposition 4 leads to $\mu Q^-(v,\beta)^k \preccurlyeq \nu Q(u,\beta)^k$ for every $k = 1, 2, \ldots$. Letting $k \to \infty$ and noting that each row of $\lim_{k\to\infty} Q^-(v,\beta)^k$ and $\lim_{k\to\infty} Q(u,\beta)^k$ is identical to $\mu^-(v,\beta)$ and $\mu(u,\beta)$, respectively, we obtain $\mu^-(v,\beta) \preccurlyeq \mu(u,\beta)$. A similar argument shows $\mu(u,\beta) \preccurlyeq \mu^+(v,\beta)$. □

7. More general forms of interaction

We have so far considered the log-linear dynamic on a strategic form game with a fixed set of players. Our results in previous sections also hold under more general forms of interactions among players specified by a finite graph which includes as a special case the local interactions studied in [4].

Recall that our basic game is a finite I-person game G in strategic form, $(S_i, u_i)_{i=1,\ldots,I}$. Each $i = 1, \ldots, I$ will now be referred to as a *player position*. A local interaction over a basic I-person game G is described by a multipartite graph $\mathcal{G} = (N_1, \ldots, N_I, E)$ where $N_i \neq \varnothing$, $N_i \cap N_j = \varnothing$ for any i and $j \neq i$, and $E \subset N \equiv N_1 \times \cdots \times N_I$. Each element of N_i represents an individual who plays a role of player i in G. Each element in E (edges) represents a profile of I players, one from each N_i, who interact to play G. We assume that every individual in every N_i participates to play G at least once. That is, the projection of E to the i-th coordinate is N_i itself. For each i and $n \in N_i$, let $E_{-i}(n)$ be the collection of $I-1$ players who interact with an individual n. That is, $E_{-i}(n) = \{e_{-i} \in N_{-i} \mid (n, e_{-i}) \in E\}$ where $N_{-i} = \bigtimes_{j\neq i} N_j$.

The local interaction game $G(\mathcal{G})$ has $N_1 \cup \cdots \cup N_I$ as the set of players and each player $n \in N_i$ has the set of actions S_i.[21] Players' choices of action are summarized by a *configuration* $\phi = (\phi_1, \ldots, \phi_I)$ where $\phi_i : N_i \to S_i$ represents choices of strategies by players in N_i. Given a configuration ϕ, the payoff to player $n \in N_i$ is

$$U_n(\phi) = \sum_{e_{-i}\in E_{-i}(n)} u_i\big(\phi_i(n), \phi_{-i}(e_{-i})\big) \tag{7.1}$$

where $\phi_{-i}(e_{-i}) = (\phi_j(e_j))_{j\neq i}$.

The state space of the log-linear dynamic on $G(\mathcal{G})$ is the set of all configurations, $\Phi = S_1^{N_1} \times \cdots \times S_I^{N_I}$. As before, at each stage, given a current state ϕ, one player n from one position i (i.e., $n \in N_i$) is given an opportunity to revise his action according to the log-linear choice rule $p_i(\cdot \mid \phi : U_n, \beta)$ defined as in (2.1) and (2.2). The probability of transition from a state ϕ to ϕ' is defined analogously to (2.3). The resulting Markov chain is again irreducible and aperiodic.

Population games studied in [4] is a special case of the local interaction defined above where $E = N_1 \times \cdots \times N_I$. Note that, if the interacting set of players are chosen randomly at each

[21] As a special case, in [4], a local interaction over a two-player symmetric game is modeled by an undirected graph (N, E) where $E \subset N \times N$, E is disjoint from the diagonal, and every node is connected to at least one other node. The payoff to $n \in N$ at a configuration $\phi : N \to S$ is given by $g_n(\phi) = \sum_{m\in N} \mathbf{I}((n,m) \in E)u(\phi(n), \phi(m))$. It is easy to convert this setup to a bipartite graph (N_1, N_2, E).

stage according to the uniform distribution over $N_1 \times \cdots \times N_I$, then the expected payoffs to each player would be a constant multiple of (7.1). Thus the local interaction model above encompasses a version of random matching.[22]

Potential games. Suppose that G is a potential game. If $v : S \to \mathbb{R}$ is a potential function for G, then it is easy to verify that $V : \Phi \to \mathbb{R}$ defined by

$$V(\phi) = \frac{1}{I} \sum_{e \in E} v\big(\phi(e)\big) = \frac{1}{I} \sum_{i=1}^{I} \sum_{n \in N_i} \sum_{e_{-i} \in E_{-i}(n)} v\big(\phi_i(n), \phi_{-i}(e_{-i})\big) \tag{7.2}$$

is a potential function for $G(\mathcal{G})$. Clearly, V is maximized at any configuration $\phi = (\phi_1, \ldots, \phi_I)$ such that for every edge $e = (e_1, \ldots, e_I) \in E$, the strategy profile $\phi(e) = (\phi_1(e_1), \ldots, \phi_I(e_I))$ maximizes v. Let Φ^* be the set of such configurations. By Theorem 1 we have

Corollary 3. *(See [4].) Suppose that G is a potential game with a potential function v and hence $G(\mathcal{G})$ is a potential game with a potential function V defined by (7.2). Then the invariant distribution of the log-linear dynamic on $G(\mathcal{G})$ with parameter β is*

$$\mu_\phi(V, \beta) = \frac{e^{\beta V(\phi)}}{\sum_{\phi' \in \Phi} e^{\beta V(\phi')}} \tag{7.3}$$

and the set of stochastically stable states is Φ^*.

Precise form of Φ^* depends on the detail of interaction, i.e., the graph $\mathcal{G}$. But if there is a unique strategy profile s^* that maximizes a potential function v for the basic game G, then Φ^* is a singleton set $\{\phi^*\}$ where $\phi^*(e) = s^*$ for every $e \in E$.

Local potential. From the discussion of the potential games above it is clear that if s^* is a local potential maximizer of the underlying game G with a local potential function v, then ϕ^* where $\phi^*(e) = s^*$ for every $e \in E$ is a local potential maximizer of the local interaction game $G(\mathcal{G})$ with a local potential V defined by (7.2). The next result thus follows from Theorem 2.

Corollary 4. *Suppose that s^* is a local potential maximizer in G with a local potential function $v : S \to \mathbb{R}$. If each u_i is supermodular, or v is supermodular, then ϕ^* is the unique stochastically stable state of the log-linear dynamic on $G(\mathcal{G})$.*

8. Concluding remarks

Let us make a few remarks concerning the literature on equilibrium selection that employ local potential and related notions. First, a weighted LP-max (Definition 3) is robust to incomplete information under the same conditions as ours together with the following additional condition: the local potential function or the payoff functions must exhibit diminishing marginal returns (see [14, Proposition 3]). Under the same condition, a weighted LP-max is globally accessible (for all sufficiently small degrees of friction) and linearly absorbing under the perfect-foresight dynamic (see [18, Lemma 4.2, Corollary 4.2]).

[22] The model can also accommodate assortative matching, but the stochastic stability of a potential and a local potential maximizer is not guaranteed due to the cardinality issue discussed in Section 5.

As to a weighted LP-max (Definition 3) with weights that may be player-specific but do not depend on strategies, which includes the versions of LP-max used in our main theorem as well as the one used in the results of Section 5, the above results hold without the condition of diminishing marginal returns. This is because this version of LP-max is a monotone potential maximizer as defined in [14], and a monotone potential maximizer is robust to incomplete information (see [14]) as well as linearly absorbing and globally accessible (see [18]) without this condition.[23]

A simple case of monotone potential maximizer can be described as follows. A strategy profile s^* is a monotone potential maximizer if one can find (a) a total order on each player's strategy set with s_i^* as the maximum and (b) a real valued function v defined on the set of strategy profiles with two conditions. First, s^* uniquely maximizes the value of v. Second, against any conjecture on other players' strategies, a player has an action that is a best response in the given game and is at least as large in the order found in (a) as some best response action in the game with v as the payoff function common to all players.

In contrast to the log-linear dynamic, robustness and stability concepts used in [14] and [18] only depend on the best response correspondence and so in particular are invariant to an affine transformation of payoffs.

We end with a comment on our proof method: comparison of Markov chains. This technique may be useful for characterizing the long-run behavior of dynamics other than the log-linear dynamic (e.g. [9,25]). If the invariant distribution of a dynamic is known for a game with payoffs $u' = (u_1,', \ldots, u'_I)$, and if $u = (u_1, \ldots, u_I)$ and u' are ordered in some appropriate sense – e.g., under the log-linear dynamics and $u' = v$ is a local potential for u – then one could infer the invariant distribution for payoff functions u. Though this statement remains at an intuitive level for now, attempts to formalize this general idea seem worthwhile.

Acknowledgments

This research was initiated while the authors were members of School of Social Science at Institute for Advanced Study, Princeton, USA, during 2007–2008 academic year. Okada gratefully acknowledges the support at the institute through Richard B, Fisher membership. Tercieux likewise acknowledges the support through Deutsche Bank membership. The authors thank the participants of Economic Theory Workshop at Institute for Advanced Study (Princeton), GRIPS Workshop on Global Games (Tokyo), Paris Workshop on Game Theory, and Microeconomic Theory Seminar at Princeton University and at Rutgers University. Helpful conversations and comments from Sylvain Chassang, Richard P. McLean, Stephen Morris, Daisuke Oyama, and Marcin Peski are gratefully acknowledged. We are especially thankful to Satoru Takahashi for careful readings and helpful suggestions at various stages of this research. We are indebted to the associate editor and two anonymous referees for suggestions that have greatly improved the paper. All remaining errors are, of course, solely by the authors.

Appendix A. Proof of Proposition 1

Suppose that we have a total order $\leqslant_i$ on each S_i, $i = 1, \ldots, I$, with the property that $\Lambda(s^0, s^1, \ldots, s^L) < 0$ for every monotonic path of unilateral deviations $s^0 = s^*, s^1, \ldots, s^L \neq s^*$.

---

[23] In general, a weighted LP-max is a monotone potential maximizer if the diminishing marginal returns condition is satisfied. Hence, the results in the above paragraph also follows from those on monotone potential maximizers.

We will verify that the function v defined by (3.4) is an appropriate local potential. It is clear from (3.4) that $v(s^*) > v(s)$ for every $s \neq s^*$. It remains to show that v satisfies (b-1) and (b-2) in Definition 2.

Fix a player i and $s_{-i} \in S_{-i}$. To verify (b-1) take $s_i \leqslant_i s_i^*$ where $s_i \neq s^*$. Let $s^0 = s^*, s^1, \dots, s^L = (s_i^+, s_{-i})$ be a path that attains max Λ among all monotonic paths of unilateral deviations starting at s^* and ending at (s_i^+, s_{-i}). Thus, $v(s_i^+, s_{-i}) = \Lambda(s^0, s^1, \dots, s^L)$. Consider a path of unilateral deviations $s^0 = s^*, s^1, \dots, s^L = (s_i^+, s_{-i}), s^{L+1} = (s_i, s_{-i})$ which is monotonic as $s_i \leqslant_i s_i^+ \leqslant_i s_i^*$ and $s_i \neq s_i^+$. By the definition of v we have $v(s_i, s_{-i}) \geqslant \Lambda(s^0, s^1, \dots, s^L, s^{L+1})$ and so

$$\begin{aligned} v\big(s_i^+, s_{-i}\big) - v(s_i, s_{-i}) &\leqslant \Lambda\big(s^0, s^1, \dots, s^L\big) - \Lambda\big(s^0, s^1, \dots, s^L, s^{L+1}\big) \\ &= u_i\big(s_i^+, s_{-i}\big) - u_i(s_i, s_{-i}), \end{aligned}$$

which proves (b-1). The argument for (b-2) is similar.

Conversely, assume that s^* is an LP-max with a local potential function v. If $s^0 = s^*, s^1, \dots, s^L \neq s^*$ is a monotonic path of unilateral deviations (w.r.t. the orders associated with s^*), then it is straightforward to see that repeated applications of Definition 2 (2)-(b-1) and (b-2) lead to $u_{i_\ell}(s^\ell) - u_{i_\ell}(s^{\ell-1}) \leqslant v(s^\ell) - v(s^{\ell-1})$ for each $\ell = 1, \dots, L$. Summing over ℓ we have

$$\Lambda\big(s^0, s^1, \dots, s^L\big) \leqslant v\big(s^L\big) - v\big(s^0\big) = v\big(s^L\big) - v\big(s^*\big) < 0.$$

This completes the proof. □

Appendix B. More general definition of local potential maximizer

Total orders on each strategy set is replaced by an *ordered domain*. An ordered domain on S consists of, for each $i = 1, \dots, I$, a partition of S_i, denoted by $\{S_{i1}, \dots, S_{iK_i}\}$, and a partial order $\leqslant_i$ on S_i where $s_i \leqslant_i s_i'$ if $s_i = s_i'$ or $s_i \in S_{ik}$ and $s_i' \in S_{ik'}$ with $k < k'$. We write $s_i <_i s_i'$ in the latter case. Let $\mathcal{S}_i$ be the algebra on S_i generated by $\{S_{i1}, \dots, S_{iK_i}\}$.

For each collection of I integers $\boldsymbol{k} = (k_1, \dots, k_I)$ with $0 \leqslant k_i \leqslant K_i$, we let $S_{\boldsymbol{k}} = S_{1k_1} \times \dots \times S_{Ik_I}$ and call a set of this form a measurable rectangle. Clearly, the family of measurable rectangles form a partition of S. Let $\mathcal{S}$ be the algebra on S generated by this family.

Definition 4. A set $S^* \subset S$ is a *local potential maximizer* (LP-max) for the payoff functions $u = (u_1, \dots, u_I)$ if there exist

(1) an ordered domain on S: $\{S_{i1}, \dots, S_{iK_i}\}, \leqslant_i$, $i = 1, \dots, I$, such that S^* is a measurable rectangle, i.e., $S^* = S_{\boldsymbol{k}^*}$ for some $\boldsymbol{k}^* = (k_1^*, \dots, k_I^*)$,
(2) an $\mathcal{S}$-measurable function $v : S \to \mathbb{R}$ (local potential function) such that
 (a) $\operatorname{argmax} v = S^*$,
 (b) for every i and every $s_{-i} \in S_{-i}$,
  (b-1) if $k < k_i^*$ and $s_i \in S_{ik}$, then $v(s_i^+, s_{-i}) - v(s_i, s_{-i}) \leqslant u_i(s_i^+, s_{-i}) - u_i(s_i, s_{-i})$ for all $s_i^+ \in S_{ik+1}$,
  (b-2) if $k_i^* < k$ and $s_i \in S_{ik}$, then $v(s_i, s_{-i}) - v(s_i^-, s_{-i}) \geqslant u_i(s_i, s_{-i}) - u_i(s_i^-, s_{-i})$ for all $s_i^- \in S_{ik-1}$.

A characterization of LP-max similar to Proposition 1 is as follows. Fix an ordered domain over S, $\{S_{i1}, \dots, S_{iK_i}\}, \leqslant_i$, $i = 1, \dots, I$. Recall that indices for measurable rectangles are denoted by bold letters, e.g., $\boldsymbol{k} = (k_1, \ldots, k_I)$. We will also use notations such as $(k_i', \boldsymbol{k}_{-i})$ in the usual manner. For any pair of indices $\boldsymbol{k} = (k_1, \ldots, k_I)$ and $\boldsymbol{k}'$ such that $\boldsymbol{k}' = (k_i', \boldsymbol{k}_{-i})$ for some i and $k_i' \neq k_i$, we define

$$\lambda(\boldsymbol{k}, \boldsymbol{k}') = \max\bigl[u_i(s_i', s_{-i}) - u_i(s_i, s_{-i})\bigr] \tag{B.1}$$

where max is taken over all $s_i \in S_{ik_i}$, $s_i' \in S_{ik_i'}$ and $s_{-i} \in S_{\boldsymbol{k}_{-i}}$. Thus $\lambda(\boldsymbol{k}, \boldsymbol{k}')$ is the largest payoff difference to player i when deviating unilaterally from some strategy profile in a rectangle $S_{\boldsymbol{k}}$ to another in $S_{\boldsymbol{k}'}$. We say that a (finite) sequence of indices, $\boldsymbol{k}^0, \boldsymbol{k}^1, \ldots, \boldsymbol{k}^L$, is a *monotonic path of unilateral deviations* if (i) $\boldsymbol{k}^\ell = (k_i^\ell, \boldsymbol{k}_{-i}^{\ell-1})$ with $k_i^\ell \neq k_i^{\ell-1}$ for some i for each $\ell = 1, \ldots, L$, and (ii) $k_i^0 \geqslant k_i^1 \geqslant \cdots \geqslant k_i^L$ or $k_i^0 \leqslant k_i^1 \leqslant \cdots \leqslant k_i^L$ for all i. For a monotonic path of unilateral deviations $\boldsymbol{k}^0, \boldsymbol{k}^1, \ldots, \boldsymbol{k}^L$ we set

$$\Lambda(\boldsymbol{k}^0, \boldsymbol{k}^1, \ldots, \boldsymbol{k}^L) = \sum_{\ell=1}^{L} \lambda(\boldsymbol{k}^{\ell-1}, \boldsymbol{k}^\ell). \tag{B.2}$$

Proposition 5. *A set $S^* \subset S$ is a local potential maximizer for $u = (u_1, \ldots, u_I)$ if, and only if,* (a) *there exists an ordered domain on S, $\{S_{i1}, \ldots, S_{iK_i}\}$, $\leqslant_i$, $i = 1, \ldots, I$, such that S^* is a measurable rectangle, $S^* = S_{\boldsymbol{k}^*}$, and* (b) *$\Lambda(\boldsymbol{k}^0, \boldsymbol{k}^1, \ldots, \boldsymbol{k}^L) < 0$ for every monotonic path of unilateral deviations $\boldsymbol{k}^0 = \boldsymbol{k}^*, \boldsymbol{k}^1, \ldots, \boldsymbol{k}^L \neq \boldsymbol{k}^*$. In addition, under* (a) *and* (b)*, the function $v : S \to \mathbb{R}$ defined by the formula below serves as a local potential function*:

$$v(s) = \begin{cases} 0 & \text{if } s \in S_{\boldsymbol{k}^*}, \\ \max \Lambda(\boldsymbol{k}^0, \boldsymbol{k}^1, \ldots, \boldsymbol{k}^L) & \text{if } s \in S_{\boldsymbol{k}},\ \boldsymbol{k} \neq \boldsymbol{k}^*, \end{cases} \tag{B.3}$$

where max *is taken over all monotonic paths of unilateral deviations starting at $\boldsymbol{k}^0 = \boldsymbol{k}^*$ and ending at $\boldsymbol{k}^L = \boldsymbol{k}$.*

The main result corresponding to Theorem 2 is as follows. Proofs and examples for this generalized version of LP-max can be found in [17].

Theorem 3. *Suppose that $S_{\boldsymbol{k}^*}$ is a local potential maximizer for $u = (u_1, \ldots, u_I)$ with a local potential function $v : S \to \mathbb{R}$. If each u_i is supermodular, or v is supermodular, then $s \in S$ is stochastically stable only if $s \in S_{\boldsymbol{k}^*}$, i.e., the support of* $\lim_{\beta\to\infty} \mu(u, \beta)$ *is contained in $S_{\boldsymbol{k}^*}$.*[24]

References

[1] Carlos Alós-Ferrer, Nick Netzer, The logit-response dynamics, Games Econ. Behav. 68 (2) (2010) 413–427.

[2] James Bergin, Barton L. Lipman, Evolution with state-dependent mutations, Econometrica 64 (4) (1996) 943–956.

[3] Lawrence E. Blume, The statistical mechanics of strategic interaction, Games Econ. Behav. 5 (1993) 387–424.

[4] Lawrence E. Blume, Population games, in: B. Arthur, et al. (Eds.), The Economy as an Evolving Complex System, II, Addison–Wesley, Reading, MA, 1997, pp. 425–460.

[5] David M. Frankel, Stephen Morris, Ady Pauzner, Equilibrium selection in global games with strategic complementarities, J. Econ. Theory 108 (2003) 1–44.

[24] To be clear, the supermodularity of u_i or v stated in the theorem is with respect to the partial orders on S_i associated with an ordered domain which makes $S_{\boldsymbol{k}^*}$ an LP-max. So, for example, each u_i is supermodular if, for all i, $u_i(s_i', s_{-i}) - u_i(s_i, s_{-i}) \leqslant u_i(s_i', s_{-i}') - u_i(s_i, s_{-i}')$ whenever $s_i <_i s_i'$ (i.e., $s_i \in S_{ik}$, $s_i' \in S_{ik'}$ with $k < k'$) and $s_{-i} <_{-i} s_{-i}'$ (i.e., $s_{-i} \in S_{k_{-i}}$, $s_{-i}' \in S_{k_{-i}'}$ with $k_j \leqslant_j k_j'$ for all $j \neq i$ and $k_j <_j k_j'$ for some $j \neq i$).

[6] Mark Freidlin, Alexander Wentzell, Random Perturbations of Dynamical Systems, Springer-Verlag, New York, 1984.
[7] Atsushi Kajii, Stephen Morris, The robustness of equilibria to incomplete information, Econometrica 65 (6) (1997) 1283–1309.
[8] T. Kamae, U. Urengel, G.L. O'Brien, Stochastic inequalities on partitially ordered spaces, Ann. Probab. 5 (6) (1977) 899–912.
[9] Michihiro Kandori, George J. Mailath, Rafael Rob, Learning, mutation, and long-run equilibria in games, Econometrica 61 (1993) 29–56.
[10] Jason R. Marden, S. Jeff Shamma, Revisiting log-linear learning: Asynchrony, completeness and payoff-based implementation, Games Econ. Behav., forthcoming.
[11] William A. Massey, Stochastic orderings for Markov processes on partially ordered spaces, Math. Oper. Res. 12 (2) (1987) 350–367.
[12] Dov Monderer, Lloyd S. Shapley, Potential games, Games Econ. Behav. 14 (1996) 124–143.
[13] Stephen Morris, Potential methods in interaction games, 1999.
[14] Stephen Morris, Takashi Ui, Generalized potential and robust sets of equilibria, J. Econ. Theory 124 (2005) 45–78.
[15] Alfred Müller, Comparison Methods for Stochastic Models and Risks, John Wiley & Sons, New York, 2002.
[16] Roger B. Myerson, Refinements of the Nash equilibrium concept, Int. J. Game Theory 15 (1978) 133–154.
[17] Daijiro Okada, Olivier Tercieux, Log-linear dynamics and local potential, Economics Working Paper No. 85, Institute for Advanced Study, Princeton, 2008.
[18] Daisuke Oyama, Satoru Takahashi, Josef Hofbauer, Monotone methods for equilibrium selection under perfect foresight dynamics, Theoretical Economics 3 (2) (2008) 155–192.
[19] Daisuke Oyama, Olivier Tercieux, Iterated potential and the robustness of equilibria, J. Econ. Theory 144 (4) (2009) 1726–1769.
[20] Robert Shostak, Deciding linear inequalities by computing loop residues, J. Assoc. Comput. Mach. 28 (1981) 769–779.
[21] Hal L. Smith, Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems, Math. Surv. Monogr., vol. 41, American Mathematical Society, Providence, RI, 1995.
[22] Dietrich Stoyan, Comparison Methods for Queues and other Stochastic Models, John Wiley & Sons, New York, 1984.
[23] Takashi Ui, Robust equilibria of potential games, Econometrica 69 (2001) 1373–1380.
[24] Hiroshi Uno, Nested potentials and robust equilibria, Discussion Paper 2011/9, CORE, Université catholique de Louvain, 2011.
[25] H. Peyton Young, The evolution of conventions, Econometrica 61 (1) (1993) 57–84.