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A characterization of stochastically stable networks

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Abstract Jackson and Watts (J Econ Theory 71: 44–74, 2002) have examined the dynamic formation and stochastic evolution of networks. We provide a refinement of pairwise stability, p-pairwise stability, which allows us to characterize the stochastically stable networks without requiring the "tree construction" and the computation of resistance that may be quite complex. When a $\frac{1}{2}$-pairwise stable network exists, it is unique and it coincides with the unique stochastically stable network. To solve the inexistence problem of p-pairwise stable networks, we define its set-valued extension with the notion of p-pairwise stable set. The $\frac{1}{2}$-pairwise stable set exists and is unique. Any stochastically stable networks is included in the $\frac{1}{2}$-pairwise stable set. Thus, any network outside the $\frac{1}{2}$-pairwise stable set must be considered as a non-robust network. We also show that the $\frac{1}{2}$-pairwise stable set can contain no pairwise stable network and we provide examples where a set of networks is more "stable" than a pairwise stable network.

Keywords Network formation · Pairwise stability · Stochastic stability

JEL Classification C70 · D20

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1 Introduction

The organization of individual agents into networks and groups or coalitions has an important role in the determination of the outcome of many social and economic interactions.[1] There are many possible approaches to model network formation. One is simply to model it explicitly as a non-cooperative game (see e.g. Aumann and Myerson 1988). A different approach is to analyze the networks that one might expect to emerge in the long run and to examine a sort of stability requirement that individuals not benefit from altering the structure of the network. This is the approach that was taken by Jackson and Wolinsky (1996) when defining pairwise stable networks. A network is pairwise stable if no player benefits from severing one of their links and no other two players benefit from adding a link between them, with one benefiting strictly and the other at least weakly. Another approach is to analyze the process of network formation in a dynamic framework.[2] Jackson and Watts (2002) have proposed a dynamic process in which individuals form and sever links based on the improvement that the resulting network offers them relative to the current network. This deterministic dynamic process may end at stable networks or in some cases may cycle. To explore whether some networks might be regarded as more reasonable than others, Jackson and Watts (2002) add to this deterministic process random perturbations and examine the distribution over networks as the level of random perturbations vanishes.

Exploiting the tree construction of Freidlin and Wentzel (1984), Jackson and Watts (2002) have shown that the outcome of their selection process (called stochastically stable networks) can be fully characterized in terms of resistances. However, these results are not always helpful in determining the outcome, because the required computation for resistances and the tree construction may be quite complex.[3] Thus we do not have much knowledge on which network will arise in these processes in general. In order to extend the applicability of these results, more succinct criteria are needed to determine the outcome of this selection theory. One goal of the paper is to find a criterion for network selection that is free from the computation of resistances and the tree construction.[4]

[1] Jackson (2003, 2005) has provided a survey of models of network formation.

[2] Watts (2001) has extended the Jackson and Wolinsky model to a dynamic process but she has limited attention to the specific contest of the connections model and a particular deterministic dynamic.

[3] To be more precise, this problem is known to be NP-complete in complexity theory, see Garey and Johson (1979, p 206). We know that for NP-complete problems, all *known* algorithms to solve the problem require time which is exponential in the problem size (for instance in the number of individuals considered).

[4] In noncooperative games Young (1993), Ellison (1993) and Kandori et al. (1993) among others have applied the Freidlin and Wentzell (1984) techniques in order to provide evolutionary models that select among (strict) Nash equilibria. But these results are submitted to the same criticism than Jackson and Watts (2002) and so they are not always helpful in determining the selected action profiles. Then, some authors have looked for criteria for equilibrium (or non-equilibrium) selection that are free from the computation of resistances and the tree construction. For instance, Young

We propose a new concept, p-pairwise stability, which is a refinement of the notion of pairwise stability. A network is said to be p-pairwise stable if when we add a set of links to this network (or sever a set of links), then if we allow players to successively create or delete links, they will come back to the initial network. The parameter $p \in [0, 1]$ indicates the fraction of links that can be modified: $p = 0$ means that no link may be added or severed, $p = 1$ means that all links may be modified. Thus, 0-pairwise stability reverts to Jackson and Wolinsky (1996) pairwise stability concept. Thus, a network is said to be $\frac{1}{2}$-pairwise stable if when we add a set of links to this network (or sever a set of links) such that the number of changes is less than half the total of possible changes, then if we allow players to successively create or delete links, they will come back to the initial network without moving away from it.

We show that when a $\frac{1}{2}$-pairwise stable network exists, it is unique. Moreover it is the only stochastically stable network in Jackson and Watts (2002) stochastic evolutionary process. But while our notion of a $\frac{1}{2}$-pairwise stable network leads to a unique selection when it exists, it does not always exist. Therefore, we define its set-valued extension with the notion of $\frac{1}{2}$-pairwise stable set of networks that is proved to exist and to coincide with the $\frac{1}{2}$-pairwise stable network when it exists. We also show that if a network is stochastically stable then it belongs to the $\frac{1}{2}$-pairwise stable set of networks. Thus, any network outside the $\frac{1}{2}$-pairwise stable set must be considered as a non-robust network. Interestingly, the $\frac{1}{2}$-pairwise stable set of networks can contain no pairwise stable network. We see this as a drawback of pairwise stability, and we provide examples where a set of networks is more "stable" than a pairwise stable network.

The paper is organized as follows. In Sect. 2 we define the notion of p-pairwise stable network and we study its properties. In Sect. 3 we propose a set-valued extension, the p-pairwise stable set of networks. In Sect. 4 we provide an evolutionary foundation to the $\frac{1}{2}$-pairwise stable set of networks. In Sect. 5 we conclude.

2 *p*-Pairwise stable networks

Let $N = \{1, ..., n\}$ be the finite set of players who are connected in some network relationship. The network relationships are reciprocal and the network is thus modeled as a non-directed graph.[5] A network g is a list of players who are linked to each other. For simplicity, we denote the link between i and j by ij, so $ij \in g$ indicates that i and j are linked in the network g. Let g^N be the set of all subsets of N of size 2. The network g^N is referred to as the complete network.

(1993) has shown that in a two player, two action game, only the risk-dominant equilibrium (in the sense of Harsanyi and Selten 1988) is stochastically stable. This result was generalized by Maruta (1997) and Durieu et al. (2003) to two players finite games.

[5] Bala and Goyal (2000) have studied network formation in directed networks, see also Dutta and Jackson (2000).

The set $G^N = \{g \subseteq g^N\}$ denotes the set of all possible networks on N. [6] The network obtained by adding link ij to an existing network g is denoted $g+ij$ and the network obtained by deleting link ij from an existing network g is denoted $g-ij$. For any network g, let $N(g) = \{i \mid \exists j \text{ such that } ij \in g\}$ be the set of players who have at least one link in the network g.

The utility of a network to player i is given by a function $u_i : G^N \to \mathbb{R}$. Let u denote the vector of functions $u = (u_1, ..., u_n)$. A utility function tells us what value accrues to any given player as a function of the network. Let $v(g) = \sum_{i \in N} u_i(g)$ be the total utility in g. A network $g \in G^N$ is efficient relative to u if it maximizes $v(g)$. A network $g \in G^N$ Pareto dominates a network $g' \in G^N$ relative to u if $u_i(g) \geq u_i(g')$ for all $i \in N$, with strict inequality for at least one $i \in N$. A network $g \in G^N$ is Pareto efficient relative to u if it is not Pareto dominated and finally, a network $g \in G^N$ is Pareto dominant if it Pareto dominates any other network.

A simple way to analyze the networks that one might expect to emerge in the long run is to examine a sort of equilibrium requirement that agents not benefit from altering the structure of the network. A weak version of such condition is the pairwise stability notion defined by Jackson and Wolinsky (1996). A network is pairwise stable if no player benefits from severing one of their links and no other two players benefit from adding a link between them, with one benefiting strictly and the other at least weakly.

Definition 1 *A network g is pairwise stable with respect to u if*

(i) *for all* $ij \in g$, $u_i(g) \geq u_i(g-ij)$ *and* $u_j(g) \geq u_j(g-ij)$, *and*
(ii) *for all* $ij \notin g$, *if* $u_i(g) < u_i(g+ij)$ *then* $u_j(g) > u_j(g+ij)$.

Let us say that g' is adjacent to g if $g' = g+ij$ or $g' = g-ij$ for some ij. A network g' defeats g if either $g' = g-ij$ and $u_i(g') > u_i(g)$, or if $g' = g+ij$ with $u_i(g') \geq u_i(g)$ and $u_j(g') \geq u_j(g)$ with at least one inequality holding strictly. Hence, a network is pairwise stable if and only if it is not defeated by another (necessarily adjacent) network.

Let us define a notion of distance between two networks which will be crucial in our analysis. For $g, g' \subseteq g^N$ we denote by

$$d(g,g') \equiv \frac{\#\left\{ij \in g^N \mid (ij \in g \wedge ij \notin g') \vee (ij \notin g \wedge ij \in g')\right\}}{\# g^N}$$

the *distance* between g and g'. That is, $d(g,g')$ is the number of links that g does have while g' does not, plus the number of links that g does not have while g' does, the total being divided by the maximum number of links. Thus, $d(g,g') \in [0,1]$ is the fraction of links that differ between g and g'.

[6] Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion. We also use the symbols $\vee$ and $\wedge$ which mean "or" and "and", respectively. Finally, # will refer to the notion of cardinality.

An improving path is a sequence of networks that can emerge when players form or sever links based on the improvement the resulting network offers relative to the current network. Each network in the sequence differs by (at most) one link from the previous one. If a link is added, then the two players involved must both agree to its addition, with at least one of the two strictly benefiting from the addition of the link. If a link is deleted, then it must be that at least one of the two players involved in the link strictly benefits from its deletion. Formally, an improving path from a network g is an infinite sequence of graphs $\{g_k\}_{k\geq 1}$ with $g_1 = g$ and for any $k \geq 1$:

- In the case where g_k is defeated either:
  (i) $g_{k+1} = g_k - ij$ for some ij such that $u_i(g_k - ij) > u_i(g_k)$, or
  (ii) $g_{k+1} = g_k + ij$ for some ij such that $u_i(g_k + ij) > u_i(g_k)$ and $u_j(g_k + ij) \geq u_j(g_k)$.
- In the case where g_k is not defeated by any other network: $g_{k+1} = g_k$.

This is a variation on a definition due to Jackson and Watts (2002). If there exists an improving path $\{g_k\}_{k\geq 1}$ from g' "passing by" g (i.e., such that $g_k = g$ for some k), then we use the symbol $g' \to g$. For a given network g, let $\text{im}(g) = \{g' \subseteq g^N \mid g' \to g\}$. This is the set of networks for which there is an improving path from g' passing by g.

We say that an improving path $\{g_k\}_{k\geq 1}$ with $g_1 = g'$ goes *directly* from g' to g if (1) there exists $K \geq 1$ such that $g_k = g$ if and only if $k \geq K$; (2) for all $1 < k \leq K : d(g_k, g) < d(g_{k-1}, g)$. For any network g', we write $g' \longmapsto g$ if all improving paths from g' go *directly* to g. Thus, $g' \longmapsto g$ means that g should be the "endpoint" of any improving path if g' is its initial point, and that all improving paths from g' should go to g without moving away from g. For a given network g, let $IM(g) = \{g' \subseteq g^N \mid g' \longmapsto g\}$. Notice that for all $g, g', g \neq g'$, we have $g' \in IM(g) \Rightarrow g \notin IM(g')$.

The following example shows the main insight of the stability requirement we will introduce. In particular, the example shows that a network that is Pareto-dominant and pairwise stable can be "less stable" than another network.

Example 1 Consider a situation where four players can form links. The payoffs they obtained from the different network configurations are: for a non-empty network g, $u_i(g) = \#(g)$ if $i \in N(g)$ with $\#(g)$ being the number of links in g, $u_i(g) = 0$ if $i \notin N(g)$, and $u_i(g) = 10$ if g is the empty network. Figure 1 gives some of the network configurations. Both the empty network and the complete network are pairwise stable networks. The empty network is also the Pareto-dominant network. Suppose that at least two links are added to the empty network to form g'. Then, from g' all improving paths go directly to the complete network and none goes back to the empty network. Suppose now that at most four links are deleted from the complete network to form g''. Then, from g'' all improving paths go back directly to the complete network. Thus, we say that the empty network (while being the Pareto-dominant network) is "less stable" than the complete network.

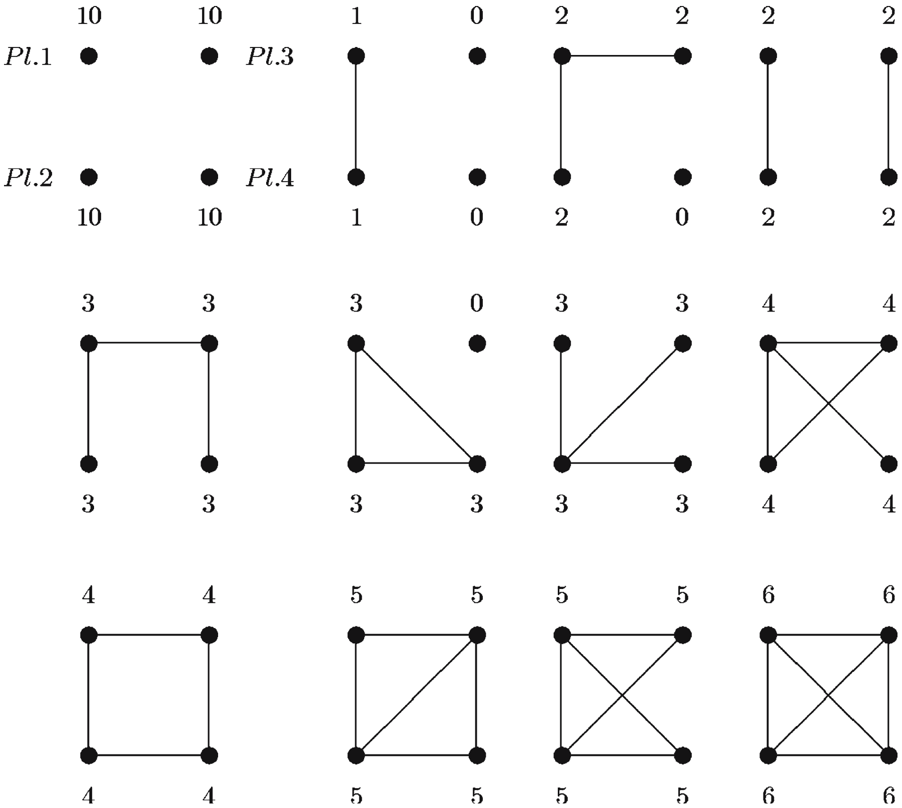

Fig. 1 The empty and complete networks are pairwise stable (Example 1)

In order to formalize our notion of stability, additional notation is needed. For any $p \in [0,1]$, let $\phi(p)$ be the smallest number larger or equal to p such that $\phi(p) \cdot \#g^N$ is an integer.

Definition 2 *Let* $p \in [0,1]$. *A network* g *is* p*-pairwise stable with respect to* u *if for all* $g' \subseteq g^N$ *such that* $d(g', g) \leq \phi(p)$, *we have* $g' \in IM(g)$.

Any network g that is p-pairwise stable is p'-pairwise stable for $p' \leq p$. The notion of p-pairwise stability is a refinement of pairwise stability in the following sense. A network g is pairwise stable if and only if it is 0-pairwise stable. Thus, any network g that is p-pairwise stable is pairwise stable.

Proposition 1 *Let* $p \geq (1/2)$. *A* p*-pairwise stable network is unique when it exists.*

Proof We proceed by contradiction. Let us assume that g^1 and g^2 are two distinct p-pairwise stable networks where $p \geq (1/2)$. Then, they are $\frac{1}{2}$-pairwise stable. If $d(g^1, g^2) \leq \phi(1/2)$, we have a straightforward contradiction. (We must have $g^1 \in IM(g^2)$ and $g^2 \in IM(g^1)$ which is not possible by definition since $g^1 \neq g^2$).

Assume now that $d(g^1, g^2) > \phi(1/2)$. Pick g^1 and delete some elements in $\{(ij \in g^1 \wedge ij \notin g^2)\}$ and add some elements in $\{(ij \notin g^1 \wedge ij \in g^2)\}$ so that the total number of changes is $\phi(1/2) \cdot \#g^N$. We obtain a network g' satisfying $d(g', g^1) = \phi(1/2)$. By construction, this network g' satisfies $d(g', g^2) \leq 1 - \phi(1/2)$. Indeed, we have that $d(g^1, g^2) \leq 1$ by definition of the distance, and from g^1 to g' we only delete links that g^1 has while g^2 does not and we only add links that g^1 does not have while g^2 does. The total number of changes being $\phi(\frac{1}{2}) \cdot \#g^N$ we have: $d(g', g^2) \leq 1 - \phi(1/2)$, and so $d(g', g^2) \leq \phi(1/2)$ because $1 - \phi(1/2) \leq \frac{1}{2} \leq \phi(\frac{1}{2})$. Then, since g^1 and g^2 are $\frac{1}{2}$-pairwise stable, we have that $g' \in IM(g^1)$, i.e. $g' \longmapsto g^1$, and $g' \in IM(g^2)$, i.e. $g' \longmapsto g^2$, which is not possible since $g^2 \neq g^1$. □

In *Example 1*, the empty network is pairwise stable and is the unique strongly stable network.[7] However, the complete network is the unique $\frac{1}{2}$-pairwise stable network. The reason is that from any network g' with $\#(g') \geq 3$ (or $d(g', g^N) \leq 1/2$) all improving paths go directly to the complete network g^N, but none goes to the empty network. [8] The next example shows that a $\frac{1}{2}$-pairwise stable network may fail to exist while a pairwise stable network exists: the unique pairwise stable is not $\frac{1}{2}$-pairwise stable because improving paths are enclosed in a cycle.

Example 2 Suppose that five players can form links. In the complete network, $u_i(g) = 8$ for all i. In any network g players $i \notin N(g)$ have a payoff $u_i(g) = 0$. In networks g such that $\#(g) \in [3, 9]$, we have $u_i(g) = 9 - \#(g)$ if $i \in N(g)$. In any g such that $\#(g) = 1$ or 2 and players 4 or 5 belong to $N(g)$ then $u_i(g) = 0$ for all i. In any g such that $\#(g) = 2$ and players 4 and 5 do not belong to $N(g)$, we have that $u_i(g) = 7$ for $i \in N(g)$. Finally, let $u_1(\{12\}) = u_3(\{13\}) = u_2(\{23\}) = 6$, $u_2(\{12\}) = u_1(\{13\}) = u_3(\{23\}) = 8$. Figure 2 presents some of these network configurations. In this example there is a unique pairwise stable network, the complete network. But, there does not exist a $\frac{1}{2}$-pairwise stable network. Indeed, from any g' such that $d(g', g^N) \geq (1/5)$, no improving path goes to g^N.

Thus, a $\frac{1}{2}$-pairwise stable network does not always exist. In the spirit of Tercieux (2006) we aim to solve the problem of non-existence of $\frac{1}{2}$-pairwise stable networks by providing a set-valued extension (a more precise connection with that approach is provided in the discussion section). Interestingly, such an approach will put into relief that a set of networks that are not pairwise stable can be more "stable" than a pairwise stable network.

[7] Jackson and van den Nouweland (2005) have introduced the notion of strongly stable networks. A strongly stable network is a network which is stable against changes in links by any coalition of individuals.

[8] Note that in all examples of the paper, we will choose the number of players N so that $\#g^N = N(N-1)/2$ is even. This will allow us to have $\phi(1/2) = 1/2$.

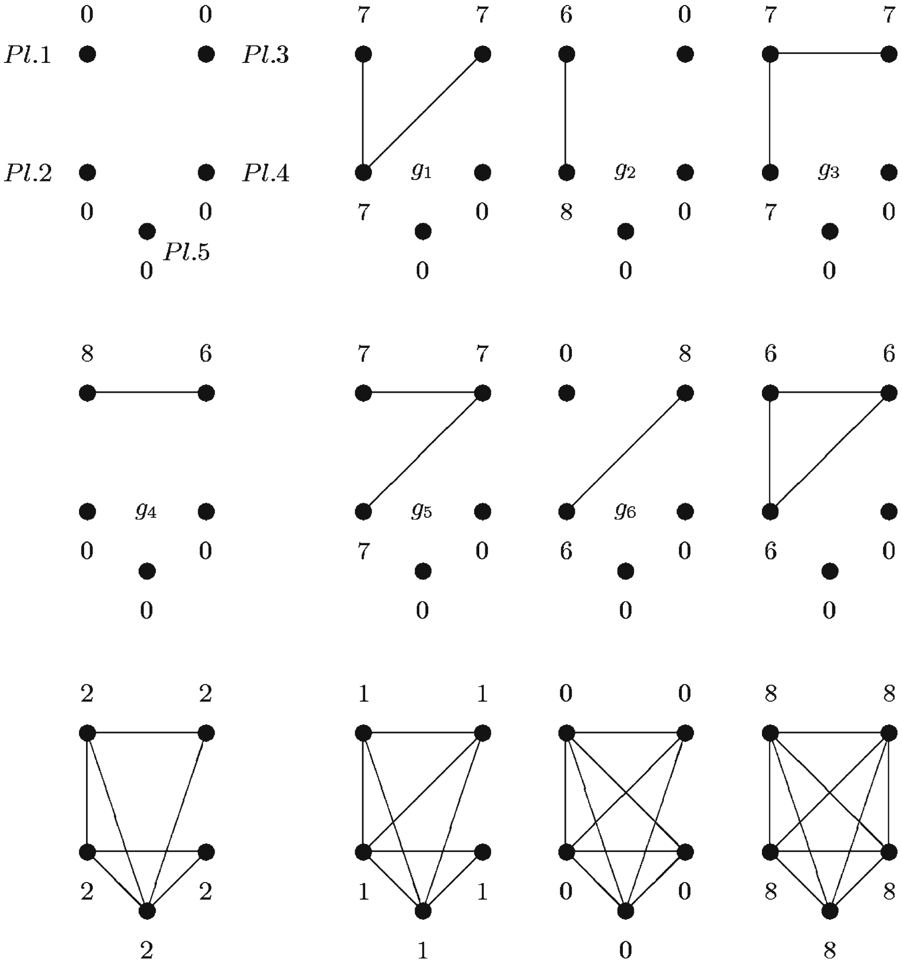

Fig. 2 An example of non-existence of $\frac{1}{2}$-pairwise stable networks (Example 2)

3 *p*-Pairwise stable sets of networks

For any two sets of networks $G, G' \subseteq G^N$, let $d(G, G') \equiv \min_{g \in G} \min_{g' \in G'} d(g, g')$. We say that an improving path $\{g_k\}_{k \geq 1}$ with $g_1 = g'$ goes *directly* from g' to $G \subseteq G^N$ if (1) there exists $K \geq 1$ such that $g_k \in G$ if and only if $k \geq K$ and (2) for all $1 < k \leq K : d(g_k, \tilde{g}) < d(g_{k-1}, \tilde{g})$ for all $\tilde{g} \in G$. For any network g', we write $g' \longmapsto G$ if all improving paths from g' go *directly* to G. For a given set of networks G, let $IM(G) = \{g' \subseteq g^N \mid g' \longmapsto G\}$.

Definition 3 *Let $p \in [0, 1]$. A set of networks $G \subseteq G^N$ is p-pairwise stable with respect to u if*

(1) *for all $g' \subseteq g^N$ such that $d(g', G) \leq \phi(p)$, we have $g' \in IM(G)$,*
(2) *there does not exist $G' \subsetneq G$ such that G' satisfies* (1).

Remark 1 The set G^N (trivially) satisfies (1) in Definition 3 for any $p \in [0, 1]$.

As underlined earlier, the main drawback of our definition of $\frac{1}{2}$-pairwise stable networks is that existence may fail also when a pairwise stable network exists. We now show that our set-valued notion of $\frac{1}{2}$-pairwise stable set always exists and is unique. As will become clear (for instance through Example 2), when there does not exist any $\frac{1}{2}$-pairwise stable network, our notion allows to eliminate many possibilities. Moreover, it is possible that the $\frac{1}{2}$-pairwise stable set of networks does not contain any pairwise stable network (see Example 2). We claim that this last point is important and underlines an important drawback of pairwise stability. The selection result we will introduce in the next section will give a foundation to this informal argument since we will prove that any network outside the $\frac{1}{2}$-pairwise stable set is not robust in a precise sense. Notice that if g is a $\frac{1}{2}$-pairwise stable network then $\{g\}$ is a $\frac{1}{2}$-pairwise stable set of networks. What our next result shows in particular is that $\{g\}$ is the only $\frac{1}{2}$-pairwise stable set of networks and thus the two notions coincide in that special case.

Proposition 2 *Let $p \geq \frac{1}{2}$. There always exists a unique p-pairwise stable set of networks.*

Proof In order to prove this result, some lemmatas are used. The first one shows that there always exists at least one p-pairwise stable set.

Lemma 1 *Let $p \in [0, 1]$. There exists at least one p-pairwise stable set of networks.*

Proof Let us proceed by contradiction. Let $p \in [0, 1]$ and assume that there does not exist any set of networks $G \subseteq G^N$ that is p-pairwise stable. This means that for G^N (see Remark 1), we can find a proper subset G^1 that satisfies (1). But again for G^1, we can find a proper subset G^2 that satisfies (1). Iterating the reasoning we can build an infinite (decreasing) sequence $\{G^k\}_{k \geq 0}$ of *distinct* elements of G^N satisfying (1). But since $\#G^N < \infty$, this is not possible; so the proof of is completed. □

The second one shows that two distinct p-pairwise stable sets cannot intersect.

Lemma 2 *Let $p \in [0, 1]$. Two (distinct) p-pairwise stable sets of networks must be disjoint.*

Proof We proceed by contradiction. Assume that G and G' are two (distinct) p-pairwise stable sets of networks and $G \cap G' \neq \emptyset$. Then, for all $g' \subseteq g^N$ such that $d(g', G \cap G') \leq \phi(p)$, we have $g' \in IM(G)$. But since this assertion is also true for G', we have that for all $g' \subseteq g^N$ such that $d(g', G \cap G') \leq \phi(p)$, $g' \in IM(G \cap G')$. Thus $G \cap G'$ satisfies (1) in Definition 3, contradicting the fact that G (and G') are p-pairwise stable sets, i.e. the minimality is violated (point (2) in Definition 3 of p-pairwise stable sets). □

By Lemma 1 we know that, for $p \geq (1/2)$, there is at least one p-pairwise stable set. We now show that, for $p \geq (1/2)$, there is a unique p-pairwise stable set. We proceed by contradiction. Assume that G^1 and G^2 are two distinct p-pairwise stable networks where $p \geq 1/2$. Then, they satisfy (1) in Definition 3 for $p = (1/2)$.

If $d(G^1, G^2) \leq \phi(1/2)$. Then we have a straightforward contradiction. (Since from some $g \in G^1$ we must have $g \in IM(G^1)$, i.e. $g \longmapsto G^1$ and $g \in IM(G^2)$, i.e. $g \longmapsto G^2$ which is not possible since $G^1 \cap G^2 = \emptyset$ by Lemma 2.)

If $d(G^1, G^2) > \phi(1/2)$, we take $g^1 (\in G^1)$ and $g^2 (\in G^2)$. Then, pick g^1 and delete some elements in $\{(ij \in g^1 \wedge ij \notin g^2)\}$ and add some elements in $\{(ij \notin g^1 \wedge ij \in g^2)\}$ so that the total number of changes is $\phi(1/2) \cdot \#g^N$. We obtain a network g' satisfying $d(g', G^1) = \phi(1/2)$. By construction, this network g' satisfies $d(g', G^2) \leq 1 - \phi(1/2) \leq (1/2) \leq \phi(1/2)$. Then, since G^1 and G^2 are p-pairwise stable for $p \geq (1/2)$ [i.e. they both satisfy (1) in Definition 3 for $p = (1/2)$], we have that $g' \in IM(G^1)$, i.e. $g' \longmapsto G^1$ and $g' \in IM(G^2)$, i.e. $g' \longmapsto G^2$ which again is not possible since $G^1 \cap G^2 = \emptyset$ by Lemma 2. □

In Example 2, the complete network is the unique pairwise stable network and there is no $\frac{1}{2}$-pairwise stable network. However, the $\frac{1}{2}$-pairwise stable set of networks is $G' = \{g_1, g_2, g_3, g_4, g_5, g_6\}$ (see Fig. 2), which does not include the complete network, because there is a cycle $g_1 \to g_2 \to g_3 \to g_4 \to g_5 \to g_6 \to g_1$ and all improving paths from any g' such that $d(g', G') \leq \frac{1}{2}$ go directly to G' and stay in G'.

We can easily link our set-valued notion to two definitions. The first one is the well-known definition of pairwise stable networks. The second one is the definition of a closed cycle due to Jackson and Watts (2002). A set of networks G, form a *cycle* if for any $g \in G$ and $g' \in G$, there exists an improving path connecting g to g'. A cycle G is a *closed cycle* if no network in G lies on an improving path leading to a network that is not in G. The proof of the first statement of Proposition 3 can be found in Appendix A, the second statement is straightforward.

Proposition 3 *G is a* 0*-pairwise stable set if and only if G is a closed cycle, and* $\{g\}$ *is a* 0*-pairwise stable set if and only if g is pairwise stable.*

4 Evolutionary selection

In this section, we show that our notion of $\frac{1}{2}$-pairwise stable networks (and $\frac{1}{2}$-pairwise stable set of networks) is relevant in the stochastic evolutionary process proposed by Jackson and Watts (2002).

4.1 The process

Let us recall first the Jackson and Watts's (2002) process. At a discrete set of times, $\{1, 2, 3, \ldots\}$ decisions to add or sever a link are made. At each date, a pair of players ij is randomly identified with probability $p(ij) > 0$. The (potential) link between these two players is the only link that can be altered at that time. If the link is already in the network, then the decision is whether to sever it, and otherwise the decision is whether to add the link. The players involved act myopically, adding the link if it makes each at least as well off and one strictly better off, and severing the link if its deletion makes either player better off. After the action is taken, there is some small probability $\varepsilon > 0$ that a mutation (or tremble, or mistake) occurs and the link is deleted if it is present, and added if it is absent.

The above process defines a (finite) Markov chain with states being the network in place at the end of a given period. Note that with mutations as part of the process, each state of the system is reachable with positive probability from every other state. The Markov chain is said to be irreducible and aperiodic, and thus has a unique corresponding stationary distribution (see Freidlin and Wentzel 1984). As ε goes to zero, the stationary distribution converges to a unique limiting stationary distribution. A network that is in the support of the limiting (as ε goes to zero) stationary distribution of the above-described Markov process is said to be *stochastically stable*. Intuitively, a stochastically stable network is one that is observed infinitely many more times than others when the probability of mutations is infinitely small. Jackson and Watts (2002) provides a characterization of stochastically stable networks using the tree construction of Freidlin and Wentzell (1984). In the following, we prove that our concept can be used to avoid this complex construction.

4.2 Relationship between p-pairwise stability and stochastic stability

The following theorem shows that under the process we have just described, the only networks that will arise with a significant frequency in the long run (i.e., the stochastically stable ones) are in the $\frac{1}{2}$-pairwise stable set.

Theorem 1 *Let G be the $\frac{1}{2}$-pairwise stable set of networks. The set of stochastically stable networks is included in G.*

The proof of Theorem 1 is given in Appendix B. This Theorem says that any network outside the $\frac{1}{2}$-pairwise stable set must be considered as a non-robust network. To be more precise, the stochastic process presented above can be thought of as a check on the robustness of pairwise networks or cycles. Although a number of networks may be pairwise stable, they can differ in how resilient they are to the random mutations. For instance, it may be relatively hard to leave and easy to get back to some networks. Our above theorem tells us that such networks are included in the $\frac{1}{2}$-pairwise stable set of networks. This result also tells us that any network that is not in the $\frac{1}{2}$-pairwise stable set is relatively easy to leave and hard to get back.

In order to understand these points, note that once the process has reached the $\frac{1}{2}$-pairwise stable set of networks G, it cannot leave it without further mutations. On the first hand, in order to get off that set, it is necessary that strictly more than ($\#g^N/2$) mutations occur (notice that in order to give the intuition of our result, we skip some technical points in assuming that N is such that ($\#g^N/2$) is an integer). If it is not the case, the process will come back to G with no further mutation. On the other hand, as it will become clear, if the process has reached a network that is outside G, it is sufficient that less than ($\#g^N/2$) mutations occur to allow the process to reach a network that belong to G. In order to see why it is so, note that from a network g' that does not belong to G, with (less than) ($\#g^N/2$) mutations, one can reach a network $\bar{g}$ such that $d(g, \bar{g}) \leq (1/2)$ where g belongs to G. Thus, by definition, the process will move to G without any further mutations. To see how we can build $\bar{g}$, we just have to add links to g' that belong to g and not to g' or to delete links that do not belong to g but belong to g'. By repeating this procedure less than ($\#g^N/2$) times, we can reach such a $\bar{g}$. Thus, there exist networks in G which are the easiest to reach from other networks, where – again – "easiest" is interpreted as requiring the fewest mutations. These networks are stochastically stable. The formal argument is given in the Appendix B.

Of course, we would like to have a full characterization of the set of stochastically stable networks. In order to do so, we provide several sufficient conditions that go in that sense. These results are corollaries of Theorem 1. The first one shows that if there exists a $\frac{1}{2}$-pairwise stable network then it must be the unique stochastically stable network. Note that this result can be seen as a parallel to the one of Young (1993) [Theorem 3, p.72] in noncooperative games.

Corollary 1 *Assume that a network g is the $\frac{1}{2}$-pairwise stable network. Then g is the unique stochastically stable network.*

The following corollary directly comes from the fact that if g is stochastically stable then g is part of a 0-pairwise stable set of networks. Furthermore, if $g \in G$ is stochastically stable and G is a 0-pairwise stable set then all $g' \in G$ are stochastically stable (this follows from Lemma 2 in Jackson and Watts (2002) together with our Proposition 3 that establishes the equivalence between a 0-pairwise stable set and a closed cycle).

Corollary 2 *Let G be the $\frac{1}{2}$-pairwise stable set of networks. If $G' \subseteq G$ is the unique 0-pairwise stable set in G then G' is the set of stochastically stable networks.*

Example 3 Suppose that three players can form links. The payoffs they obtain in any network configuration are given in Fig. 3. The complete network is the unique pairwise stable network and there is no $\frac{1}{2}$-pairwise stable network because of the cycle $g_1 \to g_2 \to g_3 \to g_4 \to g_5 \to g_6 \to g_1$. The $\frac{1}{2}$-pairwise stable set of networks is $G' = \{g_1, g_2, g_3, g_4, g_5, g_6, g^N\}$ but this set is not 0-pairwise stable. Indeed, $\{g^N\}$ is the unique 0-pairwise stable set and so by corollary 2 is the unique stochastically stable network.

The next example shows that our sufficient conditions are quite tight in the following sense: a p-pairwise stable network with $p = \frac{1}{2} - \varepsilon$ (ε small) may not be a stochastically stable network.

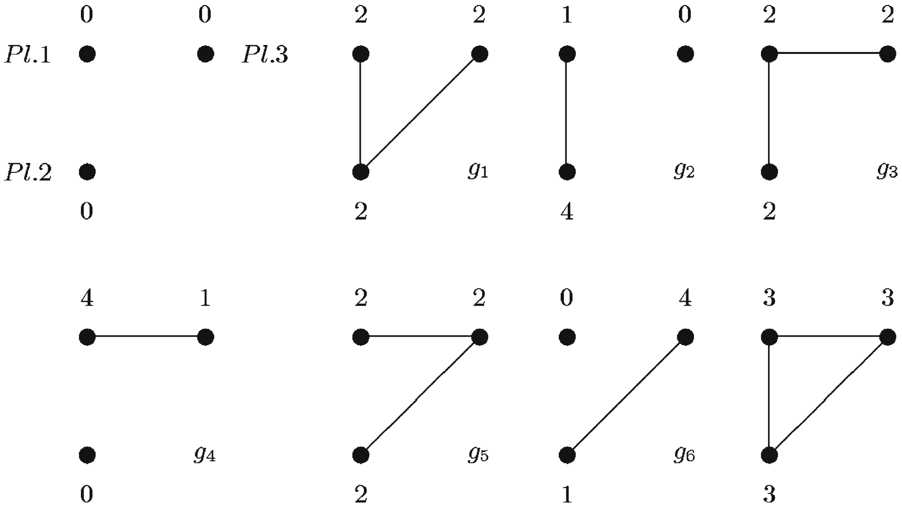

Fig. 3 $\frac{1}{2}$-pairwise stable set and stochastically stable networks (Example 3)

Example 4 Suppose that fifty players can form links. For $\#(g) \leq 611$, let $u_i(g) = 611 - \#(g)$ if $i \in N(g)$ and $u_i(g) = 0$ otherwise. For $\#(g) \geq 612$, let $u_i(g) = \#(g) - 611$ if $i \in N(g)$ and $u_i(g) = 0$ otherwise. The empty network is a p-pairwise stable network for $p \leq (610/1{,}225) \simeq 0.498$, but the empty network is not stochastically stable. The unique stochastically stable network is the complete one, which is also the unique $\frac{1}{2}$-pairwise stable network.

5 Concluding discussion

We conclude by discussing some points. First, we clarify and discuss the sense in which our main results make more simple the computations to find stochastically stable networks. Second, we clarify the cutting power of our approach. Finally, we discuss interesting parallelisms between our notion and the notions of p-dominance proposed by Morris et al. (1995) and of p-best response set due to Tercieux (2006).

5.1 Complexity and cutting power

As mentioned in Sect. 1, the tree construction problem – proposed by Jackson and Watts (2002) to find stochastically stable networks – is known to be an NP-complete problem in complexity theory.[9] While we provide simple exam-

[9] See the proof of Theorem 1 in Appendix A for a definition of the tree construction. For our purposes, a *problem* can be considered as a general question to be answered, usually possessing several parameters, or free variables, whose values are left unspecified. An *instance* of a problem is obtained by specifying particular values for all the problem parameters. If a problem is NP-complete then, there exists an instance of this problem such that all *known* algorithms to solve the problem

ples where our conditions are easier to manipulate than the one of Jackson and Watts (2002) (and so seems to be a natural first step to see if a network is stochastically stable before dealing with the explicit tree construction), we have not provided any formal argument on this. Let us conclude by a couple of clarifications on this point as well as clarifications of our contribution.

When a problem is known to be NP-complete, two approaches can be followed. (1) First of all, if a problem is NP-complete, then *there exists* one problem instance such that all known algorithm to solve the problem for this instance require time which is exponential in the problem size (hereafter, exponential time algorithm). Hence, one can try to find algorithms that would solve the problem in polynomial time for a subset of the set of possible instances. (2) One may also want to relax the problem somewhat, looking for a polynomial time algorithm that merely provides a "partial answer to the problem".

Our approach can also be divided in two lines. Firstly, we restrict our attention to *specific cases* where there exists a $\frac{1}{2}$-pairwise stable network (Corollary 1) and show that in this case, it is the unique stochastically stable network and so the tree construction can be avoided. This approach is in line of point (1) above. Secondly, we show that in the *general case*, any stochastically stable network is included in the $\frac{1}{2}$-pairwise stable set of networks (Theorem 1). This second result is in line of (2). It indeed simplifies the problem to a less ambitious one in that it requires only for a partial characterization of the set of stochastically stable networks. Doing this, one can avoid the explicit computations required by the tree construction. A formal analysis of the complexity class of our problem remains an interesting open issue left for further research.

NP-completeness is a standard notion in computer sciences, however, it is a "worst-case" measure. There are cases (as shown in Jackson and Watts 2002) where the tree construction while complicated can be achieved in a reasonable time. Thus, another way to see our results is as complementary to the tree construction proposed by Jackson and Watts (2002). Indeed, our Theorem 1 allows for a partial characterization of the set of stochastically stable networks which does not refer to the heavy computation of resistances. As shown in the examples of the paper this allows a great simplification compared to the tree construction. However, there is a cost for this since we only get a partial characterization of stochastically stable networks whereas the tree construction allows for a complete characterization. Indeed, in some cases, the $\frac{1}{2}$-pairwise stable set might be very large – and moreover larger than the set of stochastically stable networks. Thus, the simplification of the procedure leads to a loss in accuracy. Hence, as argued above, the two procedures should be seen as complementary in that it is natural to start with our simple procedure *and then* apply the tree construction only on the remaining candidates for stochastic stability.

require time which is exponential in the problem size (for instance – in our case – the number of individuals considered). See Garey and Johnson (1979) for additional details.

5.2 Relationship to p-dominance

We can draw a parallel between our concept of p-pairwise stable networks and the concept of p-dominance as introduced by Morris et al. (1995) in non-cooperative games. p-Dominance generalizes the well-known concept of risk-dominance introduced by Harsanyi and Selten (1988). A Nash equilibrium is said to be p-dominant if for any player, and any distribution of probability of this player assigning probability (at least) p to the other players playing the equilibrium, the unique best reply is the equilibrium action. In a same spirit (and because of the same inexistence problem as for p-pairwise stable networks), Tercieux (2006) introduced a set-valued extension of p-dominance with the notion of p-best response set. Briefly, a (product) set of action profiles is a p-best response set if for any player and any distribution of probability assigning probability (at least) p to the other players playing in this set, all best replies are in this set.

While having somewhat the same flavour, the concepts in the paper and the notions defined above are clearly different since the latter relies essentially on notion of probabilities while the former do not use probabilities but instead uses a notion of distance between networks. However, having this analogy in mind is – we believe – particularly useful. For instance, it allowed us to show that these concepts have many similar properties. As for p-pairwise stable networks, if a p-dominant equilibrium with $p = (1/2)$ exists, it is the unique one; and as for p-pairwise stable set of networks, a (minimal) p-best response set for $p = (1/2)$ exists and is unique. The notion of p-dominance for $p = (1/2)$ has proved to be extremely useful, for instance Maruta (1997) proved that it is selected in a stochastic best-reply dynamics *à la* Young (1993) while Kajii and Morris (1997) and Frankel et al. (2003) proved that it is selected in incomplete information frameworks.[10] Going one step further in this analogy, it would be interesting to study the stability properties of our notion in an incomplete information setting. We do not claim that our notion would be the correct one, however while our paper is a first step in this direction, trying to find the corresponding concept of risk-dominance and – more generally of p-dominance – in network formation games seems to be a promising approach.

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[10] The selection of p-dominance for $p = (1/2)$ is true in two-player games.

Appendix A. Proof of Proposition 3

In order to prove Proposition 3 that establishes the equivalence between 0-pairwise stable sets and closed cycles, we first state and prove some useful lemmata. The following lemma is stated without proof.[11]

Lemma 3 *If G is such that for all $g \in G$, $g \in IM(G)$ (note that this is* (1) *in Definition 3 of a* 0*-pairwise stable set) then there exists $C \subseteq G$ that is a closed cycle.*

The next lemma provides a first step in establishing a link between 0- pairwise stable sets and closed cycles.

Lemma 4 *If C is a closed cycle then there exists $G \subseteq C$ that is* 0*-pairwise stable.*

Proof Since C is a closed cycle, we know that for all $g \in C$, $g \in IM(C)$. Then C satisfies (1) of Definition 3 of a 0- pairwise stable set. Now assume that there does not exist any $G \subseteq C$ that is 0-pairwise stable. Then any $G \subseteq C$ has a proper subset that satisfies (1) in the definition of 0-pairwise stable sets. Now, similar to the proof of Lemma 1, this implies that there exists an infinite decreasing sequence $\{G^k\}_{k\geq 0}$ where $G^0 = C$ and $G^{k+1} \subsetneq G^k$ for all $k \geq 0$. But since $\#G^N < \infty$, this is not possible; so the proof is completed. □

Now we are ready to complete the proof of Proposition 3. We first prove the *"if" part*. Suppose that G is a closed cycle but G is not 0-pairwise stable and show that this leads to a contradiction. This last point can be due to the violation of (1) or (2) in the definition of a 0-pairwise stable set. Assume first that (1) is violated. Such a violation implies in particular that there exists $g \in G$ and $g' \notin G$ such that $g \to g'$, and it contradicts the definition of a closed cycle. Assume now that (2) is violated. This means that there exists $G' \subsetneq G$ that satisfies (1) in the definition of a 0-pairwise stable set i.e., for all $g' \in G'$, $g' \in IM(G')$. But by Lemma 3, we know that there exists a closed cycle $C \subseteq G' \subsetneq G$. Then, we have the following: first, because G is a (closed) cycle, we have that for all $g, g' \in G$, $g \to g'$. But we also have, because C is a closed cycle, that for all $g \in C(\subsetneq G)$ and $g' \in G\backslash C$, $g \to g'$ is false. Thus we obtain a contradiction.

We now prove the *"only if" part*. We know by Lemma 3 that since G is 0-pairwise stable, there exists $C \subseteq G$ that is a closed cycle. We must prove that $C = G$. So let us proceed by contradiction and assume that $C \subsetneq G$. We know by Lemma 4 that there exists $G' \subseteq C \subsetneq G$ that is 0-pairwise stable. This leads to a straightforward contradiction since it contradicts (2) (the minimality) in the 0-pairwise stability of G. This completes the proof of Proposition 3. □

Appendix B. Proof of Theorem 1

In order to prove Theorem 1, we first introduce some useful definitions and notations.

[11] A complete proof would mimic the proof of Lemma 1 in Jackson and Watts (2002, p. 273).

B.1 Definitions

For a given network g, remember that $im(g) = \{g' \subseteq g^N \mid$ there exists an improving path from g' passing by $g\}$. A path $\mathbf{p} = \{g_1, ..., g_K\}$ is a sequence of adjacent networks. The resistance of a path $\mathbf{p} = \{g_1, ..., g_K\}$ from g' to g, denoted $r(\mathbf{p})$, is computed by $r(\mathbf{p}) = \sum_{i=1}^{K-1} I(g_i, g_{i+1})$, where

$$I(g_i, g_{i+1}) = \begin{cases} 0 & \text{if } g_i \in im(g_{i+1}) \\ 1 & \text{otherwise.} \end{cases}$$

Resistance keeps track of how many mutations must occur along a special path to follow that path from one network to another. A mutation is necessary to move from one network to an adjacent one whenever it is not in the relevant player's interests to sever or add the link that distinguishes the two adjacent networks.

Let $r(g', g) = \min\{r(\mathbf{p}) \mid \mathbf{p}$ is a path from g' to $g\}$ and set $r(g, g) = 0$. Note that $r(g', g) = 0$ if and only if $g' \in im(g)$ or $g' = g$. Thus (by Proposition 3) if $g, g' \in G$ where G is 0-pairwise stable, then $r(g', g) = 0$.

Given a network g, a g-tree is a directed graph which has as vertices all networks and has a unique directed path leading from each g' to g. Let $T(g)$ denote all the g-trees, and represent a $t \in T(g)$ as a collection of ordered pairs of networks, so that $g'g'' \in t$ if and only if there is a directed edge connecting g' to g'' in the g-tree t. The resistance of a network g is computed as $r(g) = \min_{t \in T(g)} \sum_{g'g'' \in t} r(g', g'')$.

In addition as noted in Jackson and Watts (2002), only closed cycles (a pairwise stable network is a closed cycle) matter in the dynamic process. Given two closed cycles C, C', let $r(C, C') = r(g, g')$ where $g \in C$ and $g' \in C'$ and set $r(C, C) = 0$. In the sequel, the set of closed cycles will be denoted Ξ. (We will also sometimes note for a network g and a closed cycle $C' : r(g, C') = r(g, g')$ for $g' \in C'$.)

Given a closed cycle C, a C-tree is a directed graph which has as its root C, and as other vertices closed cycles and has a unique directed path from each vertex to C. Denote the set of C-trees by $T(C)$, and represent a $t \in T(C)$ as a collection of ordered pairs of networks, so that $C'C'' \in t$ if and only if there is a directed edge connecting C' to C'' in the C-tree t. The resistance of a closed cycle C is computed as $r(C) = \min_{t \in T(C)} \sum_{C'C'' \in t} r(C', C'')$.

It is well known (see Young 1993; Jackson and Watts 2002) that a network g is a stochastically stable network if and only if g belongs to a closed cycle C such that $r(C) \leq r(C')$ for all $C' \in \Xi$. We will use this characterization in order to prove our main results.

B.2 The Proof

The proof is divided into two parts:

We give a lower bound on the resistance of the transitions that begin at $g \in G$ and end at any $g' \notin G$ where $d(g', G) > \phi(1/2)$. By definition of p-pairwise stability for $p \geq (1/2)$, $r(g, g') > \phi(1/2) \cdot \#g^N \geq (\#g^N/2)$.

We give now an upper bound on the resistances of paths that begin at any $g' \notin G$ and end in G. Pick $g' \notin G$. (Note that if $d(g', G) \leq \phi(1/2)$ then, by definition of G, $g' \in IM(G)$ i.e. no mutation is necessary to go to G. Thus we will implicitly assume that $d(g', G) > \phi(1/2)$.) Picking $g \in G$, we delete some elements in $\{(ij \in g' \wedge ij \notin g)\}$ and add some elements in $\{(ij \notin g' \wedge ij \in g)\}$ so that the total number of changes is $\phi(1/2) \cdot \#g^N$. We obtain a network $\bar{g}$ satisfying $d(\bar{g}, g') = \phi(1/2)$. By construction, this network $\bar{g}$ satisfies $d(\bar{g}, g) \leq 1 - \phi(1/2) \leq \phi(1/2)$ where $g \in G$. But G is a p-pairwise stable set of networks for $p \geq (1/2)$ and so $\bar{g} \longmapsto G$. Therefore with less than $\phi(1/2) \cdot \#g^N$ mutations, we will reach a network in a closed cycle included in G (note that once the process has reached G, we cannot leave it without mutations). Therefore, $r(g', \tilde{C}) \leq \phi(1/2) \cdot \#g^N$ for some closed cycle $\tilde{C} \subset G$. Such a closed cycle will be denoted $C(g')$. Thus for every $g' \notin G$, $r(g', C(g')) \leq \phi(1/2) \cdot \#g^N$.

Suppose by contradiction that $g' \notin G$ is stochastically stable. Let C' be the closed cycle so that $g' \in C'$. First note that it must be that $d(g', G) > \phi(1/2)$. Denote by t' (one of) the C'-tree(s) ($t' \in T(C')$) that minimizes resistance. We know that there is a sequence $C_1, \ldots, C_n$ with $C_1 = C(g')(\subset G)$ and $C_n = C'$ such that $C_l C_{l+1} \in t'$ for every $l = 1, \ldots, n-1$.

In addition, there exists two closed cycles $\tilde{C}$ and $\bar{C}$ such that $\tilde{C}\bar{C} \in t'$ and $\tilde{C} \cap G \neq \emptyset$ and $\bar{C} \cap G = \emptyset$. Delete this edge and add one from C' to C_1. We obtain a tree $t'' \in T(\tilde{C})$ where $\tilde{C} \cap G \neq \emptyset$. It is easy to show that indeed, $\tilde{C} \subset G$ (because once the process has reached G it cannot go out without mutations).

By construction, $r(\tilde{C}) = r(C') - r(\tilde{C}, \bar{C}) + r(C', C_1)$. But $r(\tilde{C}, \bar{C}) = r(\tilde{g}, \bar{g})$ where $\tilde{g} \in G$ and $\bar{g} \notin G$, and so as proved above, $r(\tilde{C}, \bar{C}) > \phi(1/2) \cdot \#g^N$. In addition, $r(C', C_1) = r(g', C(g'))$ and so again as proved above, $r(C', C_1) \leq \phi(1/2) \cdot \#g^N$. Hence, $r(\tilde{C}) < r(C')$. This contradicts the fact that g' minimizes stochastic potential. This completes the proof. □

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