# Contagion in Games with Strategic Complementarities

## *Preliminary and Incomplete*

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**Abstract**

In this paper, we are interested in the notions of robustness and contagion in games with strategic complementarities. An action profile $a^\star$ of a complete information game $g$ is said to be contagious if there is a "nearby" incomplete information game (i.e., an incomplete information game with payoffs coinciding with those of $g$ with high probability) where $a^\star$ is the unique equilibrium play. On the other hand, following the definition of Kajii and Morris (1997), $a^\star$ is said to be robust to incomplete information if every "nearby" incomplete information game has an equilibrium which generates behavior close to $a^\star$. We show that in games with strategic complementarities, there always exists (at least) one contagious equilibrium. This implies that games with strategic complementarities have at most one robust equilibrium.

Finally, we establish a formal link between robustness and noise-independent selection in global games that generalizes the result of Frankel, Morris and Pauzner (2003): if an equilibrium is robust, then it is selected by the global game approach whatever the distribution of the noise is.

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# 1 Introduction

Many economic settings can be modeled as games of strategic complementarities. It is well-known that these games frequently have multiple equilibria (see for instance Takahashi (2007)). It is then natural to assess the robustness of these equilibria with respect to some given selection criterion. In this paper, focusing on the class of games with strategic complementarities, we return to the classic question of how sensitive the equilibrium predictions are to the assumption that payoffs are common knowledge.

Consider an analyst who plans to model some situation by a particular complete information game with strategic complementarities. He believes that the game describes the environment correctly with high probability. But he is also aware that the players may be facing some uncertainty about others' payoffs which they will take into account in choosing their strategies. It is then natural for him to check whether his predictions will remain good predictions when (slightly) perturbing payoffs. To be specific, fix a complete information game $\mathbf{g}$. Say that a pure Nash equilibrium $a^*$ in $\mathbf{g}$ is *contagious* if there exists a nearby incomplete information game (more precisely a sequence of nearby games that converges toward $\mathbf{g}$) with a unique (interim) rationalizable strategy profile playing $a^*$ at any state of the world[1]. Following Kajii and Morris (1997) by "nearby" incomplete information game, we mean that the sets of players and actions are the same as in $g$ and, with high probability, each player knows that his payoff function is almost the same as in $g$. We also follow Kajii and Morris (1997) and say that a Nash equilibrium is *robust to incomplete information* if in every nearby incomplete information game, there is a Bayes Nash equilibrium under which the ex ante induced action distribution is close to the equilibrium.

Our *first main result* shows that any game with strategic complementarities possesses at least one contagious equilibrium. We interpret this result as an instability

[1]The notion of contagion under complete information is discussed in Kajii and Morris (1997), Morris (1999a), Oyama and Tercieux (2005). A similar notion is used in the literature on evolutionary games: the relationship is discussed in the conclusion of this paper.

result. Indeed, as long as there are several equilibria (which again arises frequently in this class of games), there is some nearby incomplete information games in which most of them (actually: all but one) will never be played under any rationalizable strategy profile. In particular, since the contagious equilibrium we exhibit is a pure Nash equilibrium, properly mixed Nash equilibria (i.e. that are not pure Nash equilibria) are never played in some nearby incomplete information games and so are not likely to be good predictions i.e. they are not robust to incomplete information. Instability of mixed Nash equilibria in games with strategic complementarities has already been underlined using an evolutionary selection argument (see Kandori and Rob (1995), Echenique and Edlin (2004), among others). We show that a similar result obtains for a selection criterion based on incomplete information.

By definition, if there exists a robust equilibrium then there is at most one contagious equilibrium (the robust one). Hence, if the game has strategic complementarities, our result allows to show that if an equilibrium is robust, it is also the unique contagious equilibrium. In particular, this shows that in this class of games if there is a robust equilibrium it is the unique robust[2]. We know that there exists generic games with strategic complementarities which do not have any robust equilibrium (see Morris (1999b)). However, provided that there exists a robust equilibrium (for instance if the game is a potential game, see Ui (2001)), our result shows that the notion of robustness to incomplete information has a strong bite: it picks a unique equilibrium.

Our *second main result* shows a connection between the notion of robustness to incomplete information and the well-known notion of global games (see Carlsson and van Damme (1993), and Frankel, Morris and Pauzner (2001)): it shows that a robust equilibrium is actually selected by global games and so is also justified by a global game

[2]Morris and Ui (2005) lists a number of open questions about the robustness of equilibria. In particular they ask whether there exists a unique robust equilibrium given that it exists? We do not know examples of generic games with multiple robust equilibria. Our result shows that in the specific class of supermodular games, robust equilibria of generic games must be unique if they exist and so partially answers this open question.

argument. Recall that given a complete information game, the global games approach builds an incomplete information game where each player observes a noisy signal of the true payoffs. In a global game, payoffs evolve continuously along the state space and are different at each state. In addition, it is assumed that the ex ante feasible payoffs include payoffs that make the lowest and the highest actions strictly dominant. Thus, with high ex ante probability payoffs are very different from the complete information game. This is a crucial feature of global games which distinguishes the perturbation of a global game with the one of nearby incomplete information game as defined above. The main result in this literature shows that, as the noise becomes small, iterative strict dominance eliminates all equilibria but one. We say that there is noise-independent selection at some type $t$ if as the noise goes to zero, the action profile played at $t$ does not depend on the distribution of the noise. This is an important notion since the widely[3] used uniqueness result of the global game literature has bite only if the equilibrium selected does not depend on fine details of the model like the distribution of the noise.

We show that a robust equilibrium is actually selected by global games: given a type $t$ in a global game, if the complete information game associated with $t$ has a robust equilibrium then it is selected by the global game whatever the distribution of the noise is. Otherwise stated, robustness to incomplete information implies noise-independent selection. Note that this result generalizes the sufficient conditions for noise-independent selection provided in Frankel, Morris and Pauzner (2001) since the existing sufficient conditions for robustness (see Morris and Ui (2005)) are more general than the one for noise-independent selection provided in Frankel, Morris and Pauzner (2001).

To prove our main results, we show the following connection: any equilibrium that is selected in a global game for some distribution of noise at some type $t$ is contagious in the complete information game associated with $t$ (i.e. the complete information game where ex post payoffs at $t$ are common knowledge). This intermediate result is

[3]See the survey by Morris and Shin (2003).

actually the core of our argument. Our main results are derived from it.

The paper is organized as follows. Section 2 introduces formally the notions of contagion, robustness and global games. Our results are presented in Section 3. Conclusion and discussion of some extensions are provided in Section 4.

# 2 Preliminary definitions

In this section, we introduce the main notions that will be useful to present our results. Throughout our analysis, we fix the set of players $\mathcal{I} = \{1, \ldots, I\}$ and, for each player $i$, the finite and linearly ordered set of actions $A_i = \{0, 1, \ldots, n_i\}$. We write $A$ for $\Pi_{i \in \mathcal{I}} A_i$ and $A_{-i}$ for $\Pi_{j \neq i} A_j$. A complete information game is specified by, and identified with, a profile of payoff functions, $\mathbf{g} = (g_i)_{i \in \mathcal{I}}$, where $g_i : A \to \mathbb{R}$ is the payoff function of player $i$. We let $\Delta g_i(a_i \to a_i', a_{-i})$ denote the difference in utility of player $i$ from playing $a_i'$ versus $a_i$ against the action profile $a_{-i}$, i.e., $\Delta g_i(a_i \to a_i', a_{-i}) = g_i(a_i', a_{-i}) - g_i(a_i, a_{-i})$, for all actions $a_i, a_i' \in A_i$ and action profiles $a_{-i} \in A_{-i}$.

A game has strategic complementarities if for each player, whenever his opponents raise their actions, the incentive to play higher actions increases.

**Definition 1.** *The game $\mathbf{g}$ has strategic complementarities if whenever $a_i' \geq a_i$, the difference $\Delta g_i(a_i \to a_i', a_{-i})$ is nondecreasing in $a_{-i} \in A_{-i}$, i.e., $a_{-i}' \geq a_{-i}$ implies:*

$$\Delta g_i(a_i \to a_i', a_{-i}') \geq \Delta g_i(a_i \to a_i', a_{-i}).$$

## 2.1 Incomplete information games

Given a complete information game $\mathbf{g}$, we consider the following class of incomplete information games. Each player $i \in \mathcal{I}$ has a countable set of types, denoted by $T_i$. The state space is $T = \prod_{i \in \mathcal{I}} T_i$. We write $T_{-i} = \prod_{j \neq i} T_j$ and $t_{-i} = (t_j)_{j \neq i} \in T_{-i}$. For any nonempty, at most countable set $S$, we denote by $\Delta(S)$ the set of all probability distributions on $S$. Let $P \in \Delta(T)$ be the (common) prior probability distribution on $T$. We assume that $P$ satisfies $\Sigma_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) > 0$ for all $i \in \mathcal{I}$ and $t_i \in T_i$. Under this assumption, the conditional probability of $t_{-i}$ given $t_i$, $P(t_{-i}|t_i)$, is well-defined by $P(t_{-i}|t_i) = P(t_i, t_{-i})/\sum_{t'_{-i}\in T_{-i}} P(t_i, t'_{-i})$. The payoff function of player $i \in \mathcal{I}$ is a bounded function $u_i : A \times T \to \mathbb{R}$. We write $\mathbf{u}$ for the profile of functions $(u_i)_{i\in\mathcal{I}}$. We represent an incomplete information game by $(\mathbf{u}, T, P)$.

A (behavioral) strategy of player $i \in \mathcal{I}$ is a mapping $\sigma_i : T_i \to \Delta(A_i)$, where $\Delta(A_i)$ is the set of probability distributions over $A_i$. Denote by $\Sigma_i$ the set of strategies for player $i$, and let $\Sigma = \Pi_{i\in\mathcal{I}}\Sigma_i$, $\sigma = (\sigma_1, \ldots, \sigma_I) \in \Sigma$, $\Sigma_{-i} = \Pi_{j\neq i}\Sigma_j$, and $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_I) \in \Sigma_{-i}$. For a strategy $\sigma_i$, we denote by $\sigma_i(a_i|t_i)$ the probability that $a_i \in A_i$ is chosen at $t_i \in T_i$. We write $\sigma(a|t) = \Pi_{i\in\mathcal{I}}\sigma_i(a_i|t_i)$ and $\sigma_{-i}(a_{-i}|t_{-i}) = \Pi_{j\neq i}\sigma_j(a_j|t_j)$. We also write $\sigma_P(a) = \Sigma_{t\in T}P(t)\sigma(a|t)$. By a slight abuse of notation, we will sometimes write $a$ for the strategy profile $\sigma$ where the action profile $a$ is always played, i.e., where $\sigma_P(a) = 1$.

The expected payoff of player $i$ with type $t_i \in T_i$ playing $a_i \in A_i$ is:

$$U_i(a_i, \sigma_{-i})(t_i) = \sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)u_i((a_i, \sigma_{-i}(t_{-i})), (t_i, t_{-i})),$$

where $u_i((a_i, \sigma_{-i}(t_{-i})), (t_i, t_{-i})) = \sum_{a_{-i}\in A_{-i}} \sigma_{-i}(a_{-i}|t_{-i})u_i((a_i, a_{-i}), (t_i, t_{-i}))$. Let the correspondence $\mathrm{BR}_i : \Sigma_{-i} \times T_i \to A_i$ be defined for each player $i \in \mathcal{I}$ by

$$\mathrm{BR}_i(\sigma_{-i})(t_i) = \arg\max_{a_i\in A_i}\{U_i(a_i, \sigma_{-i})(t_i)\}.$$

By a slight abuse of notation, for each type $t_i \in T_i$ and for each belief $\mu_i \in \Delta(A_{-i} \times T_{-i})$ such that $\mathrm{marg}_{T_{-i}}\mu_i = P(\cdot|t_i)$, we also write $\mathrm{BR}_i(\mu_i)(t_i)$ for the set of actions $a_i \in A_i$ that maximize the expected value of $u_i(a_i, a_{-i}, t_i, t_{-i})$ under the probability distribution $\mu_i$, i.e.:

$$\mathrm{BR}_i(\mu_i)(t_i) = \arg\max_{a_i\in A_i}\{\sum_{a_{-i},t_{-i}} \mu_i(a_{-i}, t_{-i})u_i(a_i, a_{-i}, (t_i, t_{-i}))\}.$$

**Definition 2.** *A strategy profile $\sigma \in \Sigma$ is a* Bayesian Nash equilibrium *of $(\boldsymbol{u}, T, P)$ if, for each player $i \in \mathcal{I}$, all actions $a_i \in A_i$, and all types $t_i \in T_i$,*

$$\sigma_i(a_i|t_i) > 0 \Rightarrow a_i \in \mathrm{BR}_i(\sigma_{-i})(t_i).$$

We now introduce the solution concept we use for defining rationalizability. There are several notions of rationalizability for incomplete information games. The one we use, namely, interim correlated rationalizability (Battigalli (2003), Battigalli and Siniscalchi (2003) and Dekel, Fudenberg, and Morris (2007)) is the weakest one, as shown by Dekel, Fudenberg, and Morris (2007). Our results will remain valid under any stronger notion of rationalizability.

For each $i \in \mathcal{I}$, let $R_i^0[t_i] = A_i$ for all types $t_i \in T_i$. Then, for $k = 1, 2, \ldots$, for each player $i \in \mathcal{I}$ and each type $t_i \in T_i$, define $R_i^k[t_i]$ recursively by

$$R_i^k[t_i] = \left\{ a_i \in A_i \;\middle|\; \begin{array}{c} \exists \mu_i \in \Delta(T_{-i} \times A_{-i}) : \\ \mu_i(\{(t_{-i}, a_i) | a_{-i} \in R_{-i}^{k-1}[t_{-i}]\}) = 1; \\ \text{marg}_{T_{-i}} \mu_i = P(\cdot | t_i); \\ a_i \in \text{BR}_i(\mu_i) \end{array} \right\},$$

where we denote $R_{-i}^{k-1}[t_{-i}] = \prod_{j \neq i} R_j^{k-1}[t_j]$. Let $R_i^\infty[t_i] = \bigcap_{k=0}^\infty R_i^k[t_i]$.

**Definition 3.** *A strategy $\sigma_i \in \Sigma_i$ is a rationalizable strategy of player $i$ in $(\boldsymbol{u}, T, P)$ if*

$$\sigma_i(a_i | t_i) > 0 \Rightarrow a_i \in R_i^\infty[t_i]$$

*for all $a_i \in A_i$ and $t_i \in T_i$.*

We define formally the notion of nearby incomplete information game. For two given payoff functions $g_i$ and $g_i'$, we define the distance $||g_i - g_i'|| = \max_{a \in A} |g_i(a) - g_i'(a)|$. For given $\mathbf{g}$ and $\eta > 0$, consider the following subset $T_\eta^{g_i}$ of $T_i$:

$$T_\eta^{g_i} = \{t_i \in T_i : ||u_i(\cdot, (t_i, t_{-i})) - g_i|| \leq \eta, \ \forall t_{-i} \in T_{-i} \text{ with } P(t_i, t_{-i}) > 0\}.$$

When $t_i \in T_\eta^{g_i}$ is realized (and when $\eta$ is sufficiently small), player $i$ knows that his own payoffs are *approximately* given by $g_i$. We write $T_\eta^{\mathbf{g}} = \Pi_{i \in \mathcal{I}} T_\eta^{g_i}$.

**Definition 4.** *Let $\varepsilon \in [0, 1)$. The incomplete information game $(\boldsymbol{u}, T, P)$ is an $(\varepsilon, \eta)$-elaboration of the complete information game $\boldsymbol{g}$ if $P[T_\eta^{\boldsymbol{g}}] \geq 1 - \varepsilon$.*

The notion of $(\varepsilon, \eta)$-elaboration is slightly less restrictive than the notion of $\varepsilon$-elaboration defined by Kajii and Morris (1997). More precisely, an $\varepsilon$-elaboration in their sense is an $(\varepsilon, 0)$-elaboration in our sense.

**Definition 5.** *An action profile* $a^\star$ *is* contagious *in* $\boldsymbol{g}$ *if for all* $\eta > 0$ *and* $\varepsilon \in (0, 1]$, *there exists (at least) one* $(\varepsilon, \eta)$*-elaboration* $(\boldsymbol{u}, T, P)$ *of* $\boldsymbol{g}$ *where* $a^\star$ *is always played in the unique rationalizable strategy profile, i.e., where:* $R^\infty[t] = \{a^\star\}$, $\forall t \in T$.

Notice that if an action profile is contagious in **g**, then it is a Nash equilibrium in the complete information game **g**.

We now introduce the concept of robustness. We first recall the definition of a robust equilibrium due to Kajii and Morris (1997).

**Definition 6.** *An action profile* $a^\star$ *is* robust to incomplete information *in* $\boldsymbol{g}$ *if, for every* $\delta > 0$, *there exists* $\varepsilon > 0$ *such that any* $\varepsilon$*-elaboration* $(\boldsymbol{u}, T, P)$ *of* $\boldsymbol{g}$ *has a Bayesian Nash equilibrium* $\sigma$ *such that* $\sigma_P(a^\star) \geq 1 - \delta$.

If an action profile is robust to incomplete information in **g**, then it is a Nash equilibrium in the complete information game **g**. Since we define "nearby" incomplete information games by using the notion of $(\varepsilon, \eta)$-elaboration (and not that of $\varepsilon$-elaboration), our definition for robustness is slightly stronger than the one of Kajii and Morris (1997).

**Definition 7.** *An action profile* $a^\star$ *is* strictly robust to incomplete information *in* $\boldsymbol{g}$ *if, for every* $\delta > 0$, *there exists* $\varepsilon > 0$ *and* $\eta > 0$ *such that any* $(\varepsilon, \eta)$*-elaboration* $(\boldsymbol{u}, T, P)$ *of* $\boldsymbol{g}$ *has a Bayesian Nash equilibrium* $\sigma$ *such that* $\sigma_P(a^\star) \geq 1 - \delta$.

Note that whenever an equilibrium is strictly robust, it is a strict Nash equilibrium.

## 2.2 Global Games

Following the definition of Frankel, Morris and Pauzner (2003) (FMP hereafter), we now introduce formally the notion of a global game. A state $\theta$ is drawn from the real line according to a continuous density $\psi$ with support $[-1, 1]$. Each player $i$ observes a signal $x_i = \theta + \nu\zeta_i$, where $\nu > 0$ is a scale factor and each $\zeta_i$ is distributed according to a continuous density $\phi_i$ with support contained in the interval $[-\frac{1}{2}, \frac{1}{2}]$. The state variable $\theta$ and the set of variables $\{\zeta_i\}_{i \in \mathcal{I}}$ are jointly independent.

If player $i$ chooses action $a_i \in A_i$, her ex-post payoff is $v_i(a_i, a_{-i}, \theta)$ where $\theta$ is the realized state parameter and $a_{-i}$ is the action profile of $i$'s opponents. We denote by $\mathbf{v}(\cdot, \theta)$ the complete information game associated to the state $\theta$. As in FMP, we suppose that the following assumptions are satisfied.

**A 1.** [Payoff continuity] *For each player $i$ and each $a \in A$, the function $v_i(a, \cdot) : [-1, 1] \rightarrow \mathbb{R}$ is continuous.*

A player's incentive to raise her action is weakly increasing in her opponent's actions:

**A 2.** [Strategic Complementarities] *For all $\theta \in [-1, 1]$, the complete information game $\boldsymbol{v}(\cdot, \theta)$ has strategic complementarities.*

When the state is strictly higher, strictly higher actions are more appealing.

**A 3.** [State monotonicity] *There is a real number $K > 0$ such that for all $a_{-i}$, $a_i < a_i'$, and $\theta, \theta' \in [-1, 1]$ such that $\theta \leq \theta'$,*

$$\Delta v_i(a_i \rightarrow a_i', a_{-i}, \theta') - \Delta v_i(a_i \rightarrow a_i', a_{-i}, \theta) \geq K(\theta' - \theta).$$

Finally, for each player $i$, for extreme values of the payoff parameter, the extreme actions 0 and $n_i$ are strictly dominant:

**A 4.** [Dominance Regions] *There exists $d > 0$ such that, for all $\theta < -1 + d$ and for each player $i$, action 0 is strictly dominant in the complete information game $\boldsymbol{v}(\cdot, \theta)$. Symmetrically, for all $\theta > 1 - d$ action $n_i$ is strictly dominant for each player $i$.*

Since we are interested in the strategic behavior of the agents when the noise (and the scale factor $\nu$) is small, we will always assume that: $\nu < d$.

We will represent a global game by the notation $G^{\nu}(\mathbf{v}, \psi, \phi)$. The solution concept used in the global games literature is iterative strict dominance. Note that Dekel, Fudenberg and Morris (2007) have shown that the notion of iterative strict dominance is equivalent to the notion of interim correlated rationalizability, so we will use these two notions interchangeably. The following result (proved in FMP) shows that as the signal errors shrink to zero, this process selects an essentially unique Bayesian equilibrium of the game.

**Theorem 1** (FMP (2003))**.** *The global game $G^\nu(\mathbf{v}, \psi, \phi)$ has an essentially unique strategy profile surviving iterative strict dominance in the limit as $\nu \to 0$. It is an increasing pure strategy profile. More precisely, there exists an increasing pure strategy profile $s^\star_\phi$ such that if, for each $\nu > 0$, $s^\nu$ is a pure strategy profile that survives iterative strict dominance in $G^\nu(\mathbf{v}, \psi, \phi)$, then for each player $i$, $\lim_{\nu\to 0} s^\nu_i(x_i) = s^\star_{i,\phi}(x_i)$ for all $x_i$, except possibly at the finitely many discontinuities of $s^\star_{\phi,i}$.*

# 3 Main Results

In this section, we present and provide proofs for our results.

## 3.1 Contagion

**Theorem 1.** *Assume that $\boldsymbol{g}$ has strategic complementarities. Then, it has (at least) one contagious action profile.*

Thus, any game with strategic complementarities has a contagious equilibrium. This result has several implications in particular, regarding the notion of robustness to incomplete information.

The following corollaries trivially follow from Theorem 1.

**Corollary 1.** *Assume that $a^\star$ is strictly robust to incomplete information in $\boldsymbol{g}$. Then, it is contagious in $\boldsymbol{g}$ and there is no other strictly robust action profile in $\boldsymbol{g}$.*

**Corollary 2.** *No (properly) mixed Nash equilibrium can be strictly robust to incomplete information in $\boldsymbol{g}$.*

## 3.2 Global Games

We know that as the noise vanishes, a unique equilibrium is selected in a global game. However, in the general case, as FMP have shown, different ways of embedding a complete information game into global games (with different noise structures) may lead to different selections. Hence, on an operational point of view, the uniqueness result of global games seems to have bite only if the equilibrium selected is not too sensitive to the distribution of noise. This motivates the following definition.

For given $\theta \in [-1,1]$, we write $\overrightarrow{\theta}$ for the vector of signals $(x_1, \ldots, x_I)$ where $x_i = \theta$ for each player $i$.

**Definition 8.** *Let $\mathbf{v} : A \times [-1,1] \to \mathbb{R}^I$ be a profile of payoff functions satisfying A1-A4. An action profile $a^\star \in A$ is said to be noise-independently selected at some payoff parameter $\theta \in [-1,1]$ if, for all information structures $(\psi, \phi)$, the following conditions are satisfied.*

1. *The strategy profile $s^\star_\phi$ (see definition in theorem 1) is continuous at $\overrightarrow{\theta}$.*
2. *We have: $s^\star_\phi(\overrightarrow{\theta}) = a^\star$.*

The following result is an extension of FMP and Oury (2007).

**Theorem 2.** *Let $\mathbf{v} : A \times [-1,1] \to \mathbb{R}^I$ be a profile of payoff functions satisfying A1-A4. Assume that, for some $\theta \in [-1,1]$, the complete information game $\mathbf{v}(\cdot, \theta)$ has a strictly robust action profile $a^\star$. Then $a^\star$ is noise-independently selected at $\theta$.*

## 3.3 Proofs

In this subsection, we prove Theorems 1 and 2. We need the following core intermediate result whose proof is rather technical and is given in Appendix. Write $\underline{s}^\star_\phi$ ($\bar{s}^\star_\phi$) for the left (right) continuous version of $s^\star_\phi$.

**Proposition 1.** *For all $\theta \in [-1,1]$, the action profile $\underline{s}^\star_\phi(\overrightarrow{\theta})$ is contagious in the complete information game $\mathbf{v}(\cdot, \theta)$. The same result is true for the action profile $\bar{s}^\star_\phi(\overrightarrow{\theta})$.*

*Proof of Proposition 1.* See Appendix. □

In a global game, the state space is a continuum while in an $\varepsilon$- elaboration, the type space is discrete. This is why the proof of Proposition 1 involves a discretisation argument and is rather technical. In the following lines, we give an intuition of the proof.

As in FMP, we need for our proof to define a simplified notion of global game. The simplified version $G^\nu_\star(\mathbf{v},\phi)$ of the global game $G^\nu(\mathbf{v},\psi,\phi)$ is a game in which, for each $i \in \mathcal{I}$, player $i$'s prior over $\theta$ is uniform over $[-1,1]$, and player $i$'s payoff depends directly on *her signal* $x_i$ instead of the realized state $\theta$. More precisely, the payoff to player $i$ if action profile $(a_i, a_{-i})$ is chosen and she observes signal $x_i$ is given by $v_i(a_i, a_{-i}, x_i)$ if $x_i \in [-1,1]$ and $v_i(a_i, a_{-i}, -1)$ (resp. $v_i(a_i, a_{-i}, 1)$) if $x_i < -1$ (resp. $x_i > 1$). Finally, as in the global game $G^\nu(\mathbf{v},\psi,\phi)$, the noise structure in the simplified game $G^\nu_\star(\mathbf{v},\phi)$ is characterized by the profile $\phi$ of probability density functions and the noise parameter $\nu$.

The reason why FMP used the simplified game $G^\nu_\star(\mathbf{v},\phi)$ in their proofs is the following. On the one hand, as the noise errors vanish, the game $G^\nu(\mathbf{v},\psi,\phi)$ "converges" toward the simplified game $G^\nu_\star(\mathbf{v},\phi)$. Indeed, when the signal errors tend toward zero, each player is almost sure that the true payoff parameter $\theta$ is very close to her signal. Since the density $\psi$ is continuous, it is approximately constant on the small interval of values that are still possible given her signal. On the other hand, the analysis of the simplified game $G^\nu_\star(\mathbf{v},\phi)$ is much simpler than that of the global game $G^\nu(\mathbf{v},\psi,\phi)$. In particular, FMP have proved the following (intermediate) result which will be useful in the sequel.

**Lemma A 1** (FMP (2003))**.** *Let $G^\nu_\star(\mathbf{v},\phi)$ be a simplified global game. There exists a weakly increasing strategy profile $s^\nu_\phi$ such that any profile that survives iterative strict dominance in $G^\nu_\star(\mathbf{v},\phi)$ must agree with $s^\nu_\phi$ except perhaps at the finitely many discontinuities of $s^\nu_\phi$.*

Fix some arbitrarily small scale factor $\nu$ and consider the game $G^\nu_\star(\mathbf{v},\phi)$. We need some additional notations. For each player $i$, define the family of thresholds $\{\theta^\nu(a_i)\}_{a_i \in A_i}$ by

$$\theta^\nu(a_i) = \inf\{x_i | s^\nu_{\phi,i}(x_i) \geq a_i\}.$$

For each player $i$ and all actions $a_i \neq n_i$, we write $a_i^+$ for the smallest action strictly larger than $a_i$ and played under $s^\nu_{\phi,i}$. Symmetrically, for all actions $a_i \neq 0$, we write $a_i^-$

for the largest action strictly smaller than $a_i$ and played under $s^{\nu}_{\phi,i}$. (By convention, we write $n_i^+ = n_i$ and $0^- = 0$.)

Let $\tilde{\theta} \in [-1, 1]$ be such that there is no threshold signal in the interval $[\tilde{\theta} - \nu, \tilde{\theta} + \nu]$. Put differently, assume that the payoff state $\tilde{\theta}$ is such that there exists an action profile $\tilde{a}$ satisfying: $\theta^{\nu}(\tilde{a}_i) < \tilde{\theta} - \nu$ and $\theta^{\nu}(\tilde{a}_i^+) > \tilde{\theta} + \nu$, for each player $i$. Let $a_i \in A_i$ be an action played under $s^{\nu}_{\phi,i}$ and strictly larger than $\tilde{a}_i$. Since $s^{\star}_{\phi}$ is an equilibrium, when player $i$ receives signal $\theta^{\nu}(a_i)$ and faces the strategy profile $s^{\star}_{\phi,-i}$, she is *exactly indifferent* between actions $a_i$ and $a_i^-$. Recall that, since the noise is assumed to be very small, when the player $i$ receives signal $\theta^{\nu}(a_i)$, her payoffs are approximately given by the function $v_i(\cdot, \theta^{\nu}(a_i))$. Consequently, by state monotonicity (Assumption 3), if her payoffs were given by the function $v_i(\cdot, \tilde{\theta})$ (instead of $v_i(\cdot, \theta^{\nu}(a_i))$), she would *strictly* prefer at signal $\theta^{\nu}(a_i)$ action $a_i^-$ to action $a_i$. Hence, by continuity (Assumption 1) and since the action space $A$ is finite, there exists $\delta > 0$ such that for each player $i$, for all actions $a_i > \tilde{a}_i$ and for all signals $x_i \in [\theta^{\nu}(a_i), \theta^{\nu}(a_i) + \delta]$, player $i$'s best response at signal $x_i$ (if her payoffs were given by $v_i(\cdot, \tilde{\theta})$ instead of $v_i(\cdot, x_i)$) would be lower than (or equal to) $a_i^-$.

Now consider the following incomplete information game $G^{\nu}_{\star\star}(\mathbf{v}(\cdot, \tilde{\theta}), \phi)$. The state variable $\theta$ is uniformly distributed over some interval $[-1, \omega]$ where $\omega > 0$ is a (very) large real number. The noise structure is the same as in the global game $G^{\nu}_{\star}(\mathbf{v}, \psi, \phi)$: when the payoff state $\theta$ is realized, each player $i$ receives a signal $x_i = \theta + \nu\zeta_i$ (where $\zeta_i$ is distributed according to the density function $\phi_i$). The payoff structure of the game $G^{\nu}_{\star\star}(\mathbf{v}(\cdot, \tilde{\theta}), \phi)$ is defined as follows. If $\theta \in [1, \omega - 2]$, the payoffs of each player $i$ are given by the function $v_i(\cdot, \tilde{\theta})$. If $\theta \in [-1, 1] \cup [\omega - 2, \omega]$, the payoffs of player $i$ are such that action $\tilde{a}_i$ is strictly dominant. Hence, ignoring the fact that the state space in the game $G^{\nu}_{\star\star}(\mathbf{v}(\cdot, \tilde{\theta}), \phi)$ is a continuum, its payoff structure is similar to that of an $\varepsilon$-elaboration of the complete information game $\mathbf{v}(\cdot, \tilde{\theta})$ (with $\varepsilon$ small).

Write $\bar{l}^{\nu}_{\phi}$ for the following transformation of the strategy profile $s^{\nu}_{\phi}$. For each player $i$, $\bar{l}^{\nu}_{\phi,i}(x_i)$ is equal to

- $\tilde{a}_i$ if $x_i < \theta^{\nu}(\tilde{a}_i^+)$,

- $s^{\nu}_{\phi,i}(x_i)$ if $x_i \in [\theta^{\nu}(\tilde{a}_i^{+}), 1]$ , and,

- $n_i$ if $x_i > 1$.

Notice that, because of the lower dominance region, each undominated strategy profile must be lower than $\bar{l}^{\nu}_{\phi}$. In addition, recall that $\theta^{\nu}(\tilde{a}_i) < \tilde{\theta} - \nu$ and $\theta^{\nu}(\tilde{a}_i^{+}) > \tilde{\theta} + \nu$. Consequently, even if $\bar{l}^{\nu}_{\phi,i}(x_i) \neq s^{\nu}_{\phi,i}(x_i)$ when $x_i < \theta^{\nu}(\tilde{a}_i)$, the following property is by construction satisfied: for all action $a_i > \tilde{a}_i$, the distribution of action profiles faced by player $i$ at signal $\theta^{\nu}(a_i)$ under the strategy profile $\bar{l}^{\nu}_{\phi}$ in the game $G^{\nu}_{\star\star}(\mathbf{v}(\cdot, \tilde{\theta}), \phi)$ is the same as that faced by player $i$ at the same signal in the game $G^{\nu}_{\star}(\mathbf{v}, \phi)$ under the strategy profile $s^{\nu}_{\phi}$. Hence, by the above argument, there exists $\delta > 0$ such that, for each player $i$, if a strategy $s_i$ is a best response to the strategy profile $\bar{l}^{\nu}_{\phi,-i}$, we must have: $s_i(x_i) \leq \bar{l}^{\nu}_{\phi,i}(x_i - \delta)$ if $x_i > -1 + \delta$ and $s_i(x_i) = \tilde{a}_i$ otherwise.

By strategic complementarities (Assumption 2), this means that each strategy surviving two rounds of elimination of strictly dominated strategies must satisfy the same property. Iterating this argument, we prove that in each rationalizable strategy, the actions chosen by player $i$ must be smaller than $\tilde{a}_i$. A symmetric argument can be used to prove that in the unique rationalizable strategy profile of the game $G^{\nu}_{\star\star}(\mathbf{v}(\cdot, \tilde{\theta}), \phi)$, the action profile $\tilde{a}$ is always played.

Using Proposition 1, we now give the proof of Theorem 1.

*Proof of Theorem 1.* For each player $i$, we define the payoff function $v_i : A \times [-1, 1] \to \mathbb{R}$ to be such that for all $\theta \in [-1, 1]$, all actions $a_i \in \{0, 1, \ldots, n_i\}$ and all action profiles $a_{-i} \in A_{-i}$,

$$v_i(a_i, a_{-i}, \theta) = g_i(a_i, a_{-i}) + \alpha a_i \theta,$$

where $\alpha > 0$ is sufficiently large for Assumption 4 (Dominance Regions) to be satisfied. It can easily be checked that Assumptions 1, 2, and 3 are also satisfied. It is thus possible to define a global game $G^{\nu}(\mathbf{v}, \psi, \phi)$ where $\psi$ is some continuous density function with support $[-1, 1]$ and $\phi = (\phi_i)_{i \in \mathcal{I}}$ is a vector of continuous density functions with support contained in the interval $[-\frac{1}{2}, \frac{1}{2}]$. By Proposition 1, we know that $\underline{s}^{\star}_{\phi}(0)$ is contagious in the complete information game $\mathbf{g}$, which completes the proof. □

We now turn to the proof of Theorem 2.

*Proof of Theorem 2.* Since the action profile $a^\star$ is a robust equilibrium in the complete information game $\mathbf{v}(\cdot,\theta)$ , no other action profile may be contagious in $\mathbf{v}(\cdot,\theta)$. Hence, by Proposition 1, for any profile $\phi$ of density functions, we must have:

$$\underline{s}^\star_\phi(\overrightarrow{\theta}) = a^\star = \bar{s}^\star_\phi(\overrightarrow{\theta}).$$

Since the strategy profile $s^\star_\phi$ is weakly increasing, this implies in particular that $s^\star_\phi$ is continuous at $\theta^\star$ and that: $s^\star_\phi(\overrightarrow{\theta}) = a^\star$. □

# 4 Conclusion

In this paper, we proved two main results. First, the existence of a contagious equilibrium in games with strategic complementarities. Second, we prove that robustness implies noise independent selection. Here, we used the notion of contagion and robustness in incomplete information games. Closely related notions have been used in the literature on evolutionary games as well as in the one on perfect foresight dynamics (à la Matsui and Matsuyama (1995)). Connections between those notions and settings have been discussed in Morris (1999, 2000) and Takahashi (2007). We believe that our results can have counter-parts in these different settings. Let us briefly illustrate why using the notion of contagion in evolutionary settings .

Roughly, this notion of contagion is usually defined in local interaction games, i.e. games where players are placed on a network and interact only with their direct neighborhood. In those games, a time structure is assumed: at each date $t$, players choose a best reply to the average strategy played by their neighbors at date $t-1$. We say that an equilibrium $a^*$ is contagious if there exists some network structure such that whenever a small fraction of players play $a^*$, this action is eventually played by every player on the whole network. It is clear that the two notions of contagion: under incomplete information games and under local interaction games have a similar flavor. We believe that – drawing on this connection – our results have their counter-part in an evolutionary setting, for instance there should always exist a contagious (in the evolutionary sense above) equilibrium in games with strategic complementarities. This is left for further research.

# A Proof of Proposition 1

We consider the simplified global game $G^{\nu}_{\star}(\mathbf{v}, \phi)$ defined in subsection 3.3. Recall that $s^{\nu}_{\phi}$ is the (essentially) unique equilibrium of $G^{\nu}_{\star}(\mathbf{v}, \phi)$. The following lemma is the core of the proof of Proposition 1.

**Lemma 1.** *Consider a payoff parameter $\tilde{\theta} \in [-1, 1]$ and an action profile $\tilde{a} \in A$ such that for all states $\theta \in [\tilde{\theta} - \nu, \tilde{\theta} + \nu]$, we have: $s^{\nu}_{\phi}(\overrightarrow{\theta}) = \tilde{a}$. Then, the action profile $\tilde{a}$ is contagious in the complete information game $\mathbf{v}(\cdot, \tilde{\theta})$.*

*Proof of Lemma 1.* Throughout this proof, the scale factor $\nu$ is fixed. Hence, it is sometimes omitted in the superscripts.

Let $\varepsilon > 0$. To prove Lemma 1, we build an $\varepsilon$-elaboration which is similar to the game $G^{\nu}_{\star\star}(\mathbf{v}(\cdot, \tilde{\theta}), \phi)$ defined in subsection 3.3. For any positive integer $N$ and each player $i$, we define the set of types $T^N_i$ by

$$T^N_i = [-\omega^N \frac{\nu}{N}, -(\omega^N - 1)\frac{\nu}{N}, \ldots, (\omega^N - 1)\frac{\nu}{N}, \omega^N \frac{\nu}{N}],$$

where $\omega^N$ is a positive integer. We write $T^N = \prod_{i \in \mathcal{I}} T^N_i$ and, for each player $i$, $T^N = \prod_{j \neq i} T^N_j$.

We define the function $F : \mathbb{R}^I \to \mathbb{R}$ by:

$$F(x_1, \ldots, x_I) = \frac{1}{\nu^I} \int_{t \in \mathbb{R}} \prod_{i \in \mathcal{I}} \phi_i(\frac{x_i - t}{\nu}) dt.$$

For all type profiles $t \in T^N$, the prior distribution $P^N(t_1, \ldots, t_I)$ is equal to $F(t)/\beta^N$ where $\beta^N > 0$ is such that: $\sum_{t \in T^N} P^N(t) = 1$. For each player $i$, the payoff function $u_i : A \times T^N \to \mathbb{R}$ is such that:

1. Types $t_i$ in the intervals $[-\omega^N \nu/N, -\omega^N \nu/N + 2]$ and $[\omega^N \nu/N - 2, \omega^N \nu/N]$ have a dominant strategy to play action $\tilde{a}_i$;

2. For all types $t_i$ in the interval $(-\omega^N \nu/N + 2, \omega^N \nu/N - 2)$, we have:

$$u_i(\cdot, (t_{-i}, t_i)) = v_i(\cdot, \tilde{\theta}),$$

for all $t_{-i} \in T^N_{-i}$.

We fix $\omega^N$ sufficiently large for the incomplete information game $(\mathbf{u}, T^N, P^N)$ to be an $\varepsilon$-elaboration. We will show that when $N$ is sufficiently large (i.e., when the step of the discretization $\frac{\nu}{N}$ is small enough), the $\varepsilon$-elaboration $(\mathbf{u}, T^N, P^N)$ has a unique strategy profile surviving iterated deletion of strictly dominated strategies in which the action profile $\tilde{a}$ is always played. Otherwise stated, we will prove that there exists $\bar{N}$ such that, for all integers $N > \bar{N}$, $R^\infty[t] = \tilde{a}$, for all $t \in T^N$.

For the $\varepsilon$-elaboration $(\mathbf{u}, T^N, P^N)$ and for all relative integers $k$ in the interval $[-\omega^N, \omega^N]$, consider the increasing strategy profile $\bar{s}^k$ defined as follows for each player $i$:

- $\bar{s}_i^k(t_i) = \tilde{a}_i$, if $t_i < k\nu/N + 1 + \theta^\nu(\tilde{a}_i^+)$,
- $\bar{s}_i^k(t_i) = a_i$, if $t_i \in [k\nu/N + 1 + \theta^\nu(a_i), k\nu/N + 1 + \theta^\nu(a_i^+))$ for $a_i \in (\tilde{a}_i, n_i)$, and,
- $\bar{s}_i^k(t_i) = n_i$, if $t_i \geq k\nu/N + 1 + \theta^\nu(n_i)$.

By construction, for each relative integer $k \in [-\omega^N, \omega^N - 1]$: $\bar{s}^k \geq \bar{s}^{k+1}$. More precisely, we have for each player $i$ and all types $t_i > -\omega^N \frac{\nu}{N}$: $\bar{s}_i^{k+1}(t_i) = \bar{s}_i^k(t_i - \frac{\nu}{N})$. In addition, since, for each player $i$, $\theta^\nu(\tilde{a}_i^+) > -1$, action $\tilde{a}_i$ is always chosen in the strategy $\bar{s}_i^{\omega^N}$. In the sequel of the proof, we will establish that when $N$ is sufficiently large, the best response to $\bar{s}^k$ is (weakly) lower than $\bar{s}^{k+1}$ for all $k \in [-\omega^N, \omega^N - 1]$.

We first need to define some additional notations. For each $a_i \geq \tilde{a}_i$, we denote by $t_i^N(k, a_i)$ the *lowest* type of player $i$ where an action greater than or equal to $a_i$ is played in the strategy $\bar{s}_i^k$ in the $\varepsilon$-elaboration $(\mathbf{u}, T^N, P^N)$. We write $\Pr[a_{-i}|\theta^\nu(a_i)]$ for the probability of action profile $a_{-i}$ for player $i$ upon receiving the signal $\theta^\nu(a_i)$ when she faces the strategy profile $s_{\phi,-i}^\nu$ in the global game $G_\star^\nu(\mathbf{v}, \phi)$. Similarly, we write $\Pr[a_{-i}|t_i^N(k, a_i)]$ for the probability of action profile $a_{-i}$ for player $i$ upon receiving the signal $t_i^N(k, a_i)$ when she faces the strategy profile $\bar{s}_{-i}^k$ in the game $(\mathbf{u}, T^N, P^N)$.

**Claim 1.** *For all $\delta > 0$, there exists $\bar{N}(\delta)$ such that, for all $N > \bar{N}(\delta)$, all relative integers $k \in [-\omega^N, \omega^N]$, each player $i$, and each action $a_i > \tilde{a}_i$ such that $t_i^N(k, a_i) \in [-\omega^N\nu/N + 2, \omega^N\nu/N - 2]$:*

$$|\Pr[a_{-i}|t_i^N(k, a_i)] - \Pr[a_{-i}|\theta^\nu(a_i)]| < \delta, \tag{1}$$

*for all action profiles* $a_{-i} \in A_{-i}$.

*Proof of Claim 1.* We distinguish two cases. First suppose that $a_{-i}$ is not (weakly) higher than $\tilde{a}_{-i}$, i.e., that there exists $j \neq i$ such that $a_j < \tilde{a}_j$. By hypothesis, we know that for all states $\theta \in [\tilde{\theta} - \nu, \tilde{\theta} + \nu]$, $s^{\nu}_{\phi}(\theta) = \tilde{a}$. Consequently, $\theta^{\nu}(\tilde{a}_i) \leq \tilde{\theta} - \nu$ and $\theta^{\nu}(\tilde{a}_i^{+}) \geq \tilde{\theta} + \nu$ for each player $i \in \mathcal{I}$. We deduce: $\Pr[a_{-i}|\theta^{\nu}(a_i)] = 0$ for all actions $a_i > \tilde{a}_i$. On the other hand, since for all $t \in T^N$ and all integers $k \in [-\omega^N, \omega^N]$ $\bar{s}^k(t) \geq \tilde{a}$, it is by construction trivially true that: $\Pr[a_{-i}|t^N(k, a_i)] = 0$ for all actions $a_i > \tilde{a}_i$.

Now we examine the second case, i.e. we suppose that $a_{-i} \geq \tilde{a}_{-i}$. First consider the global game $G^{\nu}_{\star}(\mathbf{v}, \phi)$. Recall that when the realized state is $\theta$, for each player $i$, the conditional density function of the signal $x_i$ is given by $\frac{1}{\nu}\phi_i(\frac{x_i - \theta}{\nu})$. Because of the dominance regions in the global game $G^{\nu}_{\star}(\mathbf{v}, \phi)$ (Assumption 4), we know that $\theta^{\nu}(a_i) \in [-1 + \nu, 1 - \nu]$ for all $a_i \in A_i$. Since in the simplified game, the prior is uniformly distributed on the interval $[-1, 1]$, using Bayes' rule, we deduce that the conditional density function of type profile $x_{-i}$ for type $\theta^{\nu}(a_i)$ is given by:

$$\frac{1}{\nu^{I-1}} \frac{\int_{\theta^{\nu}(a_i)-\frac{\nu}{2}}^{\theta^{\nu}(a_i)+\frac{\nu}{2}} \phi_i(\frac{\theta^{\nu}(a_i)-\theta}{\nu}) \prod_{j \neq i} \phi_j(\frac{x_j-\theta}{\nu}) d\theta}{\int_{\theta^{\nu}(a_i)-\frac{\nu}{2}}^{\theta^{\nu}(a_i)+\frac{\nu}{2}} \phi_i(\frac{\theta^{\nu}(a_i)-\theta}{\nu}) d\theta}.$$

By the following change of variable $\varepsilon_i = \frac{\theta^{\nu}(a_i)-\theta}{\nu}$ in the denominator we obtain:

$$\int_{\theta^{\nu}(a_i)-\frac{\nu}{2}}^{\theta^{\nu}(a_i)+\frac{\nu}{2}} \phi_i(\frac{\theta^{\nu}(a_i) - \theta}{\nu}) d\theta = \nu \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi_i(\varepsilon_i) d\varepsilon_i = \nu.$$

Notice that $\phi_i(\frac{\theta^{\nu}(a_i)-\theta}{\nu}) > 0$ only if $\theta \in [\theta^{\nu}(a_i) - \frac{\nu}{2}, \theta^{\nu}(a_i) + \frac{\nu}{2}]$. Hence, we finally obtain that the conditional density function of type profile $x_{-i}$ for type $\theta^{\nu}(a_i)$ is equal to $F(x_{-i}, \theta^{\nu}(a_i))$.

For each action profile $a_{-i} \in A_{-i}$, write $D(a_{-i})$ for $\Pi_{j \neq i}[\theta^{\nu}(a_j), \theta^{\nu}(a_j^{+}))$. We now know that the probability $\Pr[a_{-i}|\theta^{\nu}(a_i)]$ of $a_{-i}$ for type $\theta^{\nu}(a_i)$ in the game $G^{\nu}_{\star}(\mathbf{v}, \phi)$ when the strategy profile $s^{\nu}_{\phi}$ is played is:

$$\int_{x_{-i} \in D(a_{-i})} F(x_{-i}, \theta^{\nu}(a_i)) dx_{-i}.$$

For all $x_{-i} \in [-1,1]^{I-1}$, let $x'_{-i}$ be defined by $x'_j = x_j - \theta^\nu(a_i)$ for all $j \neq i$. Note that we have: $F(x_{-i}, \theta^\nu(a_i)) = F(x_{-i} - \theta^\nu(a_i), 0)$. Hence, by a change in variable, we obtain:

$$\Pr[a_{-i}|\theta^\nu(a_i)] = \int_{x'_{-i} \in D(a_{-i},a_i)} F(x'_{-i}, 0) dx'_{-i},$$

where $D(a_{-i}, a_i) = \prod_{j \neq i} [\theta^\nu(a_j) - \theta^\nu(a_i), \theta^\nu(a_j^+) - \theta^\nu(a_i))$.

Now, consider the game $(\mathbf{u}, T^N, P^N)$. The ex ante probability of type $t_i^N(k, a_i)$ is

$$\sum_{t_{-i} \in T^N_{-i}} P^N(t_{-i}, t_i^N(k, a_i)).$$

For all $t_{-i} \in [-\nu, \nu]^{I-1}$, define $t'_{-i}$ by $t'_j = t_j + t_i^N(k, a_i)$ for all $j \neq i$. Since by hypothesis $t_i^N(k, a_i) \in [-\omega^N \nu/N + 2, \omega^N \nu/N - 2]$, for each $j \neq i$, we have: $t'_j \in [-\omega^N \nu/N, \omega^N \nu/N]$ and thus $t'_{-i} \in T^N_{-i}$. As a consequence, we have: $P^N(t'_{-i}, t_i^N(k, a_i)) = P^N(t_{-i}, 0)$ and the ex ante probability of type $t_i^N(k, a_i)$ can be rewritten as:

$$\sum_{t'_{-i} \in [-\nu,\nu]^{I-1}} P^N(t'_{-i}, 0).$$

Hence, using Bayes' rule, the probability $\Pr(t_{-i}|t_i^N(k, a_i))$ of $t_{-i}$ for type $t_i^N(k, a_i)$ is

$$\frac{P^N(t_{-i}, t_i^N(k, a_i))}{\sum_{t'_{-i} \in [-\nu,\nu]^{I-1}} P^N(t'_{-i}, 0)}.$$

For each action profile $a_{-i}$, the probability $\Pr[a_{-i}|t_i^N(k, a_i)]$ of $a_{-i}$ for type $t_i^N(k, a_i)$ when the strategy profile $\bar{s}^k$ is played in the incomplete information game $(\mathbf{u}, T^N, P^N)$ is thus:

$$\frac{\sum_{t_{-i} \in D^k(a_{-i})} P^N(t_{-i}, t_i^N(k, a_i))}{\sum_{t'_{-i} \in [-\nu,\nu]^{I-1}} P^N(t'_{-i}, 0)},$$

where $D^k(a_{-i}) = \Pi_{j \neq i} [t_j^N(k, a_j), t_j^N(k, a_j^+))$.

Define $t'_{-i}$ by $t'_j = t_j - t_i^N(k, a_i)$ for all $j \neq i$. By this change of variable, we finally obtain:

$$\Pr[a_{-i}|t_i^N(k, a_i)] = \frac{\sum_{t'_{-i} \in D^k(a_{-i},a_i)} P^N(t'_{-i}, 0)}{\sum_{t'_{-i} \in [-\nu,\nu]^{I-1}} P^N(t'_{-i}, 0)},$$

where $D^k(a_{-i}, a_i) = \Pi_{j\neq i}[t_j^N(k, a_j) - t_i^N(k, a_i), t_j^N(k, a_j^+) - t_i^N(k, a_i))$.

Notice that for all $t'_{-i} \in [-\nu, \nu]^{I-1}$,

$$P^N(t'_{-i}, 0) = (\frac{N}{\nu})^{I-1} \int_{x_{-i}\in L(t'_{-i})} P^N(\frac{\nu}{N}\mathrm{E}[\frac{N}{\nu}x_{-i}], 0)dx_{-i},$$

where $L(t'_{-i}) = \prod_{j\neq i}[t'_j, t'_j + \frac{\nu}{N}]$ and $\mathrm{E} : \mathbb{R}^{I-1} \to \mathbb{Z}^{I-1}$ is the integer value function, i.e., for all $x_{-i} \in \mathbb{R}^{I-1}$, $\mathrm{E}(x_{-i}) = \sup_{z_{-i}\in\mathbb{Z}^{I-1}}\{z \leq x_{-i}\}$.

As a consequence,

$$\sum_{t'_{-i}\in[-\nu,\nu]^{I-1}} P^N(t'_{-i}, 0) = (\frac{N}{\nu})^{I-1} \int_{x_{-i}\in[-\nu,\nu+\frac{\nu}{N}]^{I-1}} P^N(\frac{\nu}{N}\mathrm{E}[\frac{Nx_{-i}}{\nu}], 0)dx_{-i}.$$

And similarly,

$$\sum_{t'_{-i}\in D^k(a_{-i},a_i)} P^N(t'_{-i}, 0) = (\frac{N}{\nu})^{I-1} \int_{x_{-i}\in[-\nu,\nu+\frac{\nu}{N}]^{I-1}} \mathbf{I}_{D^k(a_{-i},a_i)} P^N(\frac{\nu}{N}\mathrm{E}[\frac{Nx_{-i}}{\nu}], 0)dx_{-i},$$

where $\mathbf{I}_{D^k(a_{-i},a_i)}$ is the indicator function of $D^k(a_{-i}, a_i)$.

Thus $\Pr[a_{-i}|t_i^N(k, a_i)]$ can be rewritten as:

$$\frac{\beta^N \int_{x_{-i}\in[-\nu,\nu+\frac{\nu}{N}]^{I-1}} \mathbf{I}_{D^k(a_{-i},a_i)} P^N(\frac{\nu}{N}\mathrm{E}[\frac{Nx_{-i}}{\nu}], 0)dx_{-i}}{\beta^N \int_{x_{-i}\in[-\nu,\nu+\frac{\nu}{N}]^{I-1}} P^N(\frac{\nu}{N}\mathrm{E}[\frac{Nx_{-i}}{\nu}], 0)dx_{-i}}. \tag{2}$$

For each player $i$, the density $\phi_i$ is continuous. As a consequence, the function $F$ is also continuous and for each $x_{-i} \in [-\nu, \nu + \frac{\nu}{N}]^{I-1}$,

$$\lim_{N\to\infty} \beta^N P^N(\frac{\nu}{N}\mathrm{E}[\frac{Nx_{-i}}{\nu}, 0]) = F(x_{-i}, 0).$$

Hence, by the dominated convergence theorem, the denominator in the formula (2) tends toward $\int_{x_{-i}\in[-\nu,\nu]^{I-1}} F(x_{-i}, 0)dx_{-i}$ as $N \to \infty$. In addition,

$$\int_{x_{-i}\in[-\nu,\nu]^{I-1}} F(x_{-i}, 0)dx_{-i} =$$
$$\frac{1}{\nu^I} \int_{-\frac{\nu}{2}}^{\frac{\nu}{2}} \phi_i(\frac{-\theta}{\nu}) \prod_{j\neq i} \int_{x_j\in[-\nu,\nu]} \phi_j(\frac{x_j-\theta}{\nu})dx_j d\theta.$$

Notice that for all $\theta \in [-\frac{\nu}{2}, \frac{\nu}{2}]$, we have: $\int_{-\nu}^{\nu} \phi_j(\frac{x_j-\theta}{\nu})dx_j = \nu$. Thus, the above expression is equal to: $\frac{1}{\nu}\int_{-\frac{\nu}{2}}^{\frac{\nu}{2}} \phi_i(\frac{x_i-\theta}{\nu})d\theta = 1$.

Regarding the numerator of the formula (2), by construction, for any integer $k$ and each player $i$, we have:

$$t_i^N(k, a_i) = \frac{\nu}{N}(k + \mathrm{E}[(1 + \theta^\nu(a_i))\frac{N}{\nu}]),$$

for all actions $a_i > \tilde{a}_i$. Consequently, for all actions $a_i, a_j > \tilde{a}_i$,

$$t_j^N(k, a_j) - t_i^N(k, a_i) = \frac{\nu}{N}(\mathrm{E}[(1 + \theta^\nu(a_j))\frac{N}{\nu}] - \mathrm{E}[(1 + \theta^\nu(a_i)\frac{N}{\nu}])$$

does not depend on $k$. Otherwise stated, $\Pr[a_{-i}|t_i^N(k, a_i)] = \Pr[a_{-i}|t_i^N(k', a_i)]$ for all integers $k$ and $k'$ such that $t_i^N(k, a_i)$ and $t_i^N(k', a_i)$ belong to the interval $[-\omega^N \frac{\nu}{N} + 2, \omega^N \frac{\nu}{N} - 2]$. Besides, $t_j^N(k, a_j) - t_i^N(k, a_i)$ tends toward $\theta^\nu(a_j) - \theta^\nu(a_i)$ as $N$ tends toward infinity.

Hence, by the dominated convergence theorem, we conclude that $\Pr[a_{-i}|t^N(k, a_i)]$ converges toward $\Pr[a_{-i}|t(a_i)]$ for all $a_{-i}$ and all $a_i > \tilde{a}_i$. Since the space of action profiles $A$ is finite, this completes the proof. □

**Claim 2.** *There exists an integer $\bar{N}$ such that for all $N > \bar{N}$ and for each player $i$,*

$$\max \mathrm{BR}_i(\bar{s}_{-i}^{k-1}) \leq \bar{s}_i^k \tag{3}$$

*for all integers $k \geq 1 - \omega^N$ in the incomplete information game $(\boldsymbol{u}, T^N, P^N)$.*

*Proof of Claim 2.* We prove this result by induction. Because of the dominance regions in the global game $G_\star^\nu(\mathbf{v}, \phi)$ (Assumption 4), we know that, for each player $i$, $\theta^\nu(n_i) < 1 - \nu$. This implies that the strategy profile $\bar{s}^{1-\omega^N}$ is such that, for each player $i$ and each type $t_i \geq (1 - \omega^N)\nu/N + 2 - \nu \geq -\omega^N \nu/N + 2$,

$$\bar{s}_i^{1-\omega^N}(t_i) = n_i.$$

Consequently, because of the dominance regions in the incomplete information game $(\mathbf{u}, T^N, P^N)$, Equation (3) is true for $k = 1 - \omega^N$. Assume that it is true at rank $k$. We prove that it is true at rank $k + 1$.

First assume that $t_i \in (t_i^N(k, a_i^-), t_i^N(k, a_i))$ for some $a_i > \tilde{a}_i$. In this case, by construction, we know that for all $i$, $a_i$, and $k$, $t_i(k, a_i) = t_i(k+1, a_i) - \frac{\nu}{N}$. Hence, $t_i \in [t_i^N(k+1, a_i^-), t_i^N(k+1, a_i)]$. Thus, $\bar{s}_i^k(t_i) = a_i^- = \bar{s}_i^{k+1}(t_i)$. Hence we get:

$$\max \mathrm{BR}_i(\bar{s}_{-i}^k)(t_i) \leq \max \mathrm{BR}_i(\bar{s}_{-i}^{k-1})(t_i) \leq a_i^- = s_i^{k+1}(t_i)$$

where the first inequality comes from strategic complementarities (Assumption 2), and so monotonicity of $BR_i(\cdot)$, the second inequality is by the inductive hypothesis and the equality by construction of $s_i^{k+1}$.

Now, assume that there exists some $a_i > \tilde{a}_i$ such that $t_i = t_i^N(k, a_i)$. If the type $t_i$ is lower than $-\omega^N \nu/N + 2$ or higher than $\omega^N \nu/N - 2$, action $\tilde{a}_i$ is strictly dominant for type $t_i$. On the other hand, assume that the type $t_i$ is in the interval $[-\omega^N \nu/N + 2, \omega^N \nu/N - 2]$. By respectively, strategic complementarities (and so monotonicity of $BR_i(\cdot)$) and the inductive hypothesis, we know that:

$$\max \mathrm{BR}_i(\bar{s}_{-i}^k(t_i)) \leq \max \mathrm{BR}(\bar{s}_{-i}^{k-1})(t_i) \leq s_i^k(t_i) = a_i$$

We will show that there exists $\bar{N}$ such that for all $N > \bar{N}$, the type $t_i$ strictly prefers action $a_i^-$ to $a_i$ in the incomplete information game $(\mathbf{u}, T^N, P^N)$ when the strategy profile $\bar{s}_{-i}^k$ is played. Otherwise stated, $\max BR_i(\bar{s}_{-i}^k)(t_i) \leq a_i^- = s_i^{k+1}(t_i)$. We will show that $\bar{N}$ can be taken independently of $a_i$, $t_i$, and $k$ and so this will complete the proof.

Type $t_i$ knows that his payoffs are given by $v_i(\cdot, \tilde{\theta})$. Using State Monotonicity (Assumption 3) (recall that the following expectations are computed assuming $s^\nu$ is played):

$$\begin{aligned} &\textstyle\sum_{a_{-i}} \Pr(a_{-i} \mid \theta^\nu(a_i)) \Delta v_i[a_i \to a_i^-, a_{-i}, \tilde{\theta}] \\ &> \textstyle\sum_{a_{-i}} \Pr(a_{-i} \mid \theta^\nu(a_i)) \Delta v_i[a_i \to a_i^-, a_{-i}, \theta^\nu(a_i)] + K(\theta^\nu(a_i) - \tilde{\theta}). \end{aligned}$$

(By hypothesis we know that $K(\theta^\nu(a_i) - \tilde{\theta}) > \nu$.)

Let us define $s_i^\nu(\theta^\nu(a_i)^+) = \min_x \{s_i(x) \mid x \geq \theta^\nu(a_i)\}$ and $s_i^\nu(\theta^\nu(a_i)^-) = \max_x \{s_i(x) \mid x \leq \theta^\nu(a_i)\}$. In the simplified global game $G_\star^\nu(\mathbf{v}, \phi)$, by payoff continuity (A1), player

$i$ is indifferent between the actions $s_i^\nu(\theta^\nu(a_i)^+)$ and $s_i^\nu(\theta^\nu(a_i)^-)$ upon receiving signal $\theta^\nu(a_i)$ when she faces the strategy profile $s_{-i}^\nu$. Hence,

$$\sum_{a_{-i}} \Pr(a_{-i} \mid \theta^\nu(a_i))\Delta v_i[a_i \to a_i^-, a_{-i}, \theta^\nu(a_i)] = 0$$

and so we obtain:

$$\sum_{a_{-i}} \Pr(a_{-i} \mid \theta^\nu(a_i))\Delta v_i[a_i \to a_i^-, a_{-i}, \tilde{\theta}] > K(\theta^\nu(a_i) - \tilde{\theta}).$$

On the other hand, by Claim 1, we know that there exists $\bar{N}$ (which does not depend on $a_i$, $t_i$ and $k$) such that for all $N > \bar{N}$,

$$\begin{aligned} &\mid \textstyle\sum_{a_{-i}} \Pr(a_{-i} \mid t_i^N(k,a_i))\Delta u_i[a_i \to a_i^-, a_{-i}, t_i^N(k,a_i)] \\ &- \textstyle\sum_{a_{-i}} \Pr(a_{-i} \mid \theta^\nu(a_i))\Delta v_i[a_i \to a_i^-, a_{-i}, \tilde{\theta}] \mid < K(\theta^\nu(a_i) - \tilde{\theta}) \end{aligned}$$

(recall that at $t_i$ player $i$ knows his own payoffs and that $u_i(\cdot, t_i^N(k,a_i)) = v_i(\cdot, \tilde{\theta})$) Thus we get $\sum_{a_{-i}} \Pr(a_{-i} \mid t_i^N(k,a_i))\Delta u_i[a_i \to a_i^-, a_{-i}, t_i^N(k,a_i)] > 0$. Hence, $a_i^-$ is strictly preferred to $a_i$. This reasoning can be applied to to any $b_i > a_i^-$, which shows that $\max BR_i(\bar{s}_{-i}^k)(t_i) \leq a_i^-$ as claimed. □

We now complete the proof of Lemma 1. Consider what happens when we iteratively delete strictly dominated strategies in the $\varepsilon$-elaboration $(\mathbf{u}, T^N, P^N)$ with $N > \bar{N}$. Because of the dominance regions in the simplified global game $G_\nu^\star(\mathbf{v}, \phi)$ (Assumption 4), we know that for all $i$, $\theta^\nu(n_i) < 1 - \nu$. Consequently, for each player $i$, and all $t_i > -\omega^N \nu/N + 2$, we have: $\bar{s}^{-\omega^N}(t_i) = n_i$. Hence, because of the lower dominance region in the $\varepsilon-$elaboration $(\mathbf{u}, T^N, P^N)$, we must have for each $i$: $s_i \leq \bar{s}_i^{-\omega^N}$ for any strategy profile $s$ which is not strictly dominated in the game $(\mathbf{u}, T^N, P^N)$. Now any strategy profile $s$ which survives 2 rounds of deletion of strictly dominated strategies must satisfy: $s_i \leq \max BR_i(\bar{s}_{-i}^{-\omega^N}) \leq \bar{s}_{-i}^{1-\omega^N}$ where the first inequality comes strategic complementarities (Assumption 2) while the second comes from Claim 2. Iteration of this reasoning yields that any strategy profile $s$ surviving $k$ rounds of deletion of strictly dominated strategies must satisfy: $s \leq \bar{s}^{(k-1)-\omega^N}$ for all integers $k \in [1, 2\omega^N + 1]$. Recall that, for all types $t_i \in T_i^N$, $\bar{s}_i^{\omega^N}(t_i) = \tilde{a}_i$. We deduce that, in each rationalizable strategy profile $s$, each player $i$ always chooses an action (weakly) lower than $\tilde{a}_i$.

A symmetric construction applies to prove that, in each rationalizable strategy profile $s$, each player $i$ always chooses an action (weakly) greater than $\tilde{a}_i$. □

Recall that $s_\phi^\star$ is the (essentially) unique strategy profile surviving iterative strict dominance in the global game $G^\nu(\mathbf{v}, \psi, \phi)$ as the scale factor $\nu$ tends toward zero.

**Lemma 2.** *Let $\theta \in [-1, 1]$ be such that the strategy profile $s_\phi^\star$ is continuous at $\overrightarrow{\theta}$. Then there exists $\nu > 0$ such that in the simplified global game $G_\star^\nu(\mathbf{v}, \phi)$, $s_\phi^\nu(\overrightarrow{\theta'}) = s_\phi^\star(\overrightarrow{\theta})$ for all $\theta' \in [\theta - \nu, \theta + \nu]$.*

*Proof of Lemma 2.* The strategy profile $s_\phi^\star$ is continuous at $\overrightarrow{\theta}$. This means that for each player $i$,

$$\theta^\star(s_{\phi,i}^\star(\theta)) < \theta < \theta^\star(s_{\phi,i}^\star(\theta)^+).$$

On the other hand, FMP have proved (Lemmas A3 and A4) that the sequence of strategy profiles $\{s_\phi^\nu\}_{\nu \in (0,\bar{\nu}]}$ converges point-wise toward the strategy profile $s_\phi^\star$. More precisely, for each player $i$ and all actions $a_i \in A_i$, the sequence of thresholds $\{\theta^\nu(a_i)\}_{\nu \in (0,\bar{\nu}]}$ converges toward the threshold $\theta^\star(a_i)$ as $\nu$ tends toward zero.

We deduce that there is a scale factor $\nu_i > 0$ such that

$$\theta^{\nu_i}(s_{\phi,i}^\star(\theta)) + \nu_i < \theta < \theta^\star(s_{\phi,i}^\star(\theta)^+) - \nu_i.$$

Since there is a finite number of players, $\nu = \min_i \nu_i > 0$ which completes the proof. □

We can now conclude the proof of Proposition 1. By Lemmas 1 and 2 above, if $\theta \in [-1, 1]$ is such that the strategy profile $s_\phi^\star$ is continuous at $\overrightarrow{\theta}$, then the action profile $s_\phi^\star(\overrightarrow{\theta})$ is contagious in the complete information game $\mathbf{v}(\cdot, \theta)$.

Assume now that the strategy profile $\underline{s}_\phi^\star$ has a discontinuity at $\overrightarrow{\theta}$. Since the number of discontinuities of $\underline{s}_\phi^\star$ is finite, for any sequence $\{x^n\}_{n\in\mathbb{N}}$ converging toward $\overrightarrow{\theta^-}$, there is an integer $\bar{n}$ such that for all $n \geq \bar{n}$: 1) $\underline{s}_\phi^\star$ is continuous at $\overrightarrow{x^n}$, and, 2) $\underline{s}_\phi^\star(\overrightarrow{x^n}) = \underline{s}_\phi^\star(\theta)$. Notice that for any given complete information game $\mathbf{g}$, if there exist an action profile $a^\star$ and a sequence $\{\mathbf{g^n}\}_{n\in\mathbb{N}}$ of complete information games converging toward $\mathbf{g}$ such that $a^\star$ is contagious in $\mathbf{g}^n$ for all integers $n \in \mathbb{N}$, then $a^\star$ is contagious in $\mathbf{g}$. As a consequence, by Assumption 1 (continuity), the action profile $\underline{s}^\star_\phi(\theta)$ is contagious in the complete information game $\mathbf{v}(\cdot, \theta)$. A similar argument applies for the right continuous version $\bar{s}^\star_\phi$ of the strategy profile $s^\star_\phi$.

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