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p-Best response set and the robustness of equilibria to incomplete information

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Abstract

In this paper, we use **p**-best response sets—a set-valued extension of **p**-dominance—in order to provide a new sufficient condition for the robustness of equilibria to incomplete information: if there exists a set S which is a **p**-best response set with $\sum_{i=1}^{I} p_i < 1$, and there exists a unique correlated equilibrium μ^* whose support is in S then μ^* is a robust Nash equilibrium.

JEL classification: C72; D82

Keywords: Set-valued concepts; Incomplete information; Refinements; Robustness; p-dominance; Risk-dominance; Higher order uncertainty

1. Introduction

Consider an analyst that captures some strategic situation by a particular complete information game. Suppose that he believes that the game describes reality with a high probability. If the analyst's prediction based on the complete information game is not qualitatively very different from some equilibrium of the real incomplete information game being played, then one will be justified in trusting his prediction. This provides a first insight of what can be a robust prediction. Robustness of Nash equilibria (hereafter NE) to incomplete information was first studied by Kajii and Morris (1997a). A NE of a complete information game is robust to incomplete information if every "nearby" incomplete information game has a (Bayesian) NE which is close to it. "Nearby" incomplete information games are such that the sets of players and actions are

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the same as in the complete information game and with high probability, each player knows that his payoffs are the same. Moreover payoffs are allowed to be different with very low probability.

Kajii and Morris (1997a) gave two sufficient conditions for the robustness of equilibria. First, if there exists a unique correlated equilibrium[1] (hereafter CE) then it is robust to incomplete information (clearly since we ensure the existence conditions of a NE and since Nash equilibria are CE, a unique CE is necessarily a unique NE). The second condition is based on the concept of **p**-dominant equilibrium. Recall that an action profile $a \equiv (a_1, \ldots, a_I)$ (we focus our attention on I-player games) is a **p**-dominant equilibrium (respectively strict **p**-dominant equilibrium) where $\mathbf{p} = (p_1, \ldots, p_I)$ if for every player i, each a_i is a best response (respectively the unique best response) to any conjecture putting probability at least p_i on other players choosing a_{-i}. Their second condition states that a **p**-dominant equilibrium with $\sum_{i=1}^{I} p_i < 1$ is a unique robust equilibrium.

The purpose of this paper is to provide a new sufficient condition for robustness using Higher Order Beliefs techniques. This condition encompasses both Kajii and Morris' (1997a) conditions and is strictly weaker. Our approach is based on the notion of **p**-best response sets introduced in Tercieux (2004). A **p**-best response set is an extension of **p**-dominance to sets of actions. A set profile $S \equiv (S_1, \ldots, S_I)$ is a **p**-best response set[2] if for every player i, for any conjecture putting probability at least p_i on other players choosing an action in S_{-i}, there exists a best response in S_i. Obviously, an action profile a is a **p**-dominant equilibrium if and only if $\{a\}$ is a **p**-best response set. Our concept can also be seen as an extension of risk-dominance introduced by Harsanyi and Selten (1988). To see why, observe that in a symmetric 2×2 game, an action profile is a $(\frac{1}{2}, \frac{1}{2})$-best response set if and only if it is risk-dominant in Harsanyi and Selten's (1988) terminology.[3] Our main result states that if there exists a set S which is a **p**-best response set with $\sum_{i=1}^{I} p_i < 1$ and there exists a unique CE μ^* whose support is in S, then μ^* is a robust NE.[4]

Our condition generalizes Kajii and Morris' (1997a) results. First, if μ^* is a unique CE, it is the only CE with support in the whole action space which is trivially a **p**-best response (for all **p**). Hence, μ^* is the unique robust Nash equilibrium which is Kajii and Morris' (1997a) first result. The second condition of Kajii and Morris (1997a) is also a particular case of our condition. Let a^* be a **p**-dominant equilibrium with $\sum_{i=1}^{I} p_i < 1$. The singleton set $\{a^*\}$ is a **p**-best response set with $\sum_{i=1}^{I} p_i < 1$ and contains a unique CE whose support is trivially in $\{a^*\}$. Then a^* is a robust NE.

We then strengthen this sufficient condition using the notion of strict **p**-best response set. A set profile $S \equiv (S_1, \ldots, S_I)$ is a strict **p**-best response set if for every player i, for any conjecture putting probability at least p_i on other players choosing an action in S_{-i}, all his best responses

1 The concept of correlated equilibria is due to Aumann (1987).

2 Note that this notion is slightly weaker than the one used in Tercieux (2004) where we require that for every player i, for any conjecture putting probability at least p_i on other players choosing an action in S_{-i}, all best responses are in S_i. As we will see later, such sets will be refered as strict **p**-best response sets.

3 Recall that in a 2×2 game, (a_1, a_2) risk-dominates (b_1, b_2) equilibrium if, $(g_1(a_1, a_2) - g_1(b_1, a_2)) \times (g_2(a_1, a_2) - g_2(a_1, b_2)) \geqslant (g_1(b_1, b_2) - g_1(a_1, b_2)) \times (g_2(b_1, b_2) - g_2(b_1, a_2))$. We say that (a_1, a_2) is the risk-dominant equilibrium. Many papers have shown that risk-dominance can be used in order to characterize stochastic best response dynamics, we have proved in Durieu et al. (2003) that this is also true for **p**-best response sets.

4 Morris and Ui (2005) investigate the robustness of set of equilibria. Our concepts could easily provide conditions on robustness of set of equilibria. More precisely, a set of correlated equilibria that have support in a **p**-best response set with $\sum_{i=1}^{I} p_i < 1$ is a robust set of equilibria.

are in S_i. Thus, an action profile a is a strict **p**-dominant equilibrium if and only if $\{a\}$ is a strict **p**-best response set. We show that if there exists a set S which is a strict **p**-best response set with $\sum_{i=1}^{I} p_i < 1$ and there exists a unique CE, μ^* whose support is in S, then μ^* is the *unique* robust NE.[5]

Together with higher order beliefs techniques, Ui (2001) and Morris and Ui (2005) have developed "potential" methods (in the manner of Monderer and Shapley, 1996) to provide new conditions for the robustness of equilibria. Morris and Ui (2005) provide a sufficient condition which is weaker than the one of Kajii and Morris (1997a) by introducing a concept of generalized potential functions. Our last section shows that our sufficient condition can be seen as a particular case of Morris and Ui's (2005) work. Nonetheless, as underlined in Morris and Ui (2005), finding generalized potential functions is a rather difficult task whereas, as shown in Section 2, **p**-best response sets can be found quite easily.

The remainder of this paper is organized as follows. Section 2 presents the concepts of **p**-best response set and strict **p**-best response set. It also links these concepts to **p**-dominance and provides two simple examples. Section 3 recalls some results on higher order beliefs and introduces the notion of robustness. Section 4 contains our main results and their proofs. Section 5 discusses the link with potential techniques.

2. p-Best response set

2.1. Definitions

Throughout our analysis, we fix a complete information game Γ consisting of a finite collection of players $\Im = \{1, \ldots, I\}$ and, for each player i, a finite action set A_i and a payoff function $g_i : A \to \mathbb{R}$, where $A = A_1 \times \cdots \times A_I$. Thus $\Gamma = [\Im, \{A_i\}_{i \in \Im}, \{g_i\}_{i \in \Im}]$. We shall denote $\prod_{j \neq i} A_j$ by A_{-i} and a generic element of A_{-i} by a_{-i}. For any finite set S, denote by $\Delta(S)$ the set of all probability measures on S. For $\mu \in \Delta(A)$, we denote by $Supp(\mu) = \{a \in A \mid \mu(a) > 0\}$ the support of μ.

Definition 1. An action distribution $\mu \in \Delta(A)$ is a correlated equilibrium of Γ if, for each $i \in \Im$, and for all $a_i, a_i' \in A_i$,

$$\sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i}) g_i(a_i, a_{-i}) \geqslant \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i}) g_i\big(a_i', a_{-i}\big).$$

Note that an action distribution $\mu \in \Delta(A)$ is a Nash equilibrium of Γ if it is a correlated equilibrium and, for all $a \in A$, $\mu(a) = \prod_{i \in \Im} \mu_i(a_i)$ where $\mu_i \in \Delta(A_i)$. This indirect way of defining Nash equilibrium is equivalent to the standard one.

Now, let $S_{-i} \subseteq A_{-i}$. In the sequel, we note $\Pi_w^{p_i}(S_{-i})$ (respectively $\Pi_s^{p_i}(S_{-i})$) for the set of distribution of probabilities that assign a probability weakly (respectively strictly) above p_i to the event that the other players play in S_{-i}. Formally:

$$\Pi_w^{p_i}(S_{-i}) = \left\{ \lambda \in \Delta(A_{-i}) \;\middle|\; \sum_{a_{-i} \in S_{-i}} \lambda(a_{-i}) \geqslant p_i \right\};$$

[5] This is also a generalization of Kajii and Morris (1997a, Corollary 5-6).

and

$$\Pi_s^{p_i}(S_{-i}) = \left\{\lambda \in \Delta(A_{-i}) \;\Big|\; \sum_{a_{-i} \in S_{-i}} \lambda(a_{-i}) > p_i\right\}.$$

Let us restate the definitions of a **p**-dominant and a strict **p**-dominant equilibrium given by Kajii and Morris (1997a).

Definition 2. Let $\mathbf{p} = (p_1, \ldots, p_I) \in [0, 1]^I$.

(i) Action profile a^* is a **p**-dominant equilibrium of Γ if for all $i \in \Im$, $a_i \in A_i$, and all $\lambda \in \Pi_w^{p_i}(a_{-i}^*)$,

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i\left(a_i^*, a_{-i}\right) \geqslant \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i(a_i, a_{-i}).$$

(ii) Action profile a^* is a strict **p**-dominant equilibrium of Γ if for all $i \in \Im$, $a_i \in A_i \backslash \{a_i^*\}$, and all $\lambda \in \Pi_s^{p_i}(a_{-i}^*)$,

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i\left(a_i^*, a_{-i}\right) > \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i(a_i, a_{-i}).$$

Next we report the definitions of **p**-best response sets and strict **p**-best response sets as defined in Tercieux (2004). These concepts have to be seen as set-valued extensions of respectively the **p**-dominant and the strict **p**-dominant concepts. A set of action profiles S is a **p**-best response set[6] if when any player i believes with probability weakly greater than p_i that the other players will play in S_{-i}, player i has at least one best reply in S_i. A set of strategy profile S is a strict **p**-best response set if when any player i believes with probability strictly greater than p_i that the other players will play in S_{-i}, all his best replies are in S_i. Formally,

Definition 3. Let $\mathbf{p} = (p_1, \ldots, p_I) \in [0, 1]^I$.

(i) For sets $S_i \subseteq A_i$, $i = 1, \ldots, I$, $S \equiv \bigtimes_{i \in \Im} S_i$ is a **p**-best response set if for all $i \in \Im$, for all $\lambda \in \Pi_w^{p_i}(S_{-i})$, there exists $a_i \in S_i$ such that

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i(a_i, a_{-i}) \geqslant \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i\left(a_i', a_{-i}\right) \quad \forall a_i' \notin S_i.$$

(ii) For sets $S_i \subseteq A_i$, $i = 1, \ldots, I$, $S \equiv \bigtimes_{i \in \Im} S_i$ is a strict **p**-best response set if for all $i \in \Im$, for all $\lambda \in \Pi_s^{p_i}(S_{-i})$, there exists $a_i \in S_i$ such that

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i(a_i, a_{-i}) > \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i\left(a_i', a_{-i}\right) \quad \forall a_i' \notin S_i.$$

Notice that if S is a **p**-best response set (respectively strict **p**-best response set) then it is also a $\mathbf{p}'$-best response set (respectively strict $\mathbf{p}'$-best response set) for any $\mathbf{p}' \geqslant \mathbf{p}$. (In the sequel, for the sake of convenience, we shall afford an abuse of notation and write $S = (S_1, \ldots, S_I)$ instead of $S = S_1 \times \cdots \times S_I$.)

[6] Related notions can be found in Basu and Weibull (1991), Börgers (1994), Kalai and Samet (1984) and Pearce (1984).

Remark 1. A is a trivial (strict) $\mathbf{p}$-best response set for any $\mathbf{p}$.

Remark 2. $(a_1^*, \dots, a_I^*)$ is a $\mathbf{p}$-dominant equilibrium (respectively strict $\mathbf{p}$-dominant equilibrium) if and only if $(\{a_1^*\}, \dots, \{a_I^*\})$ is a $\mathbf{p}$-best response set (respectively strict $\mathbf{p}$-best response set).

Remark 3. $(a_1^*, \dots, a_I^*)$ is a NE if and only if $(\{a_1^*\}, \dots, \{a_I^*\})$ is a $\mathbf{1}$-best response set.

In a companion paper Tercieux (2004) we study the notion of minimal strict $\mathbf{p}$-best response sets. More generally, one can define a minimal (strict) $\mathbf{p}$-best response set as a set $\bigtimes_{i \in \Im} S_i$ that is a (respectively strict) $\mathbf{p}$-best response set and that does not contain any proper subset that is a (respectively strict) $\mathbf{p}$-best response set. Tercieux (2004) proves the existence of minimal (strict) $\mathbf{p}$-best response sets for compact-continuous games[7] (where $\mathbf{p}$ is a symmetric vector). Using the kind of proof as in Tercieux (2004), one can prove the existence of minimal $\mathbf{p}$-best response set (where $\mathbf{p}$ is not necessarily a symmetric vector) in the same class of games. Note that in our framework where games are finite, the existence is simple for the following reason. The set of $\mathbf{p}$-best response sets is non-empty (see Remark 1), finite and partially ordered by the weak set inclusion. Thus the existence of a minimal element in that set is trivial and this is a minimal $\mathbf{p}$-best response set.

In the following, we develop two simple examples, where $\mathbf{p}$-best response sets are characterized.

2.2. Example 1

Consider the following symmetric 3×3 game. Each player has three possible actions: L, M and R.

Player 2

Player 1		L	M	R
	L	5, 5	1, 4	1, 0
	M	4, 1	3, 3	3, 3.5
	R	0, 1	3.5, 3	4, 4

There exist two (pure) Nash equilibria: (L, L) which is (p, p)-dominant for $p \geqslant \frac{2}{3}$ and (R, R) which is (p, p)-dominant for $p \geqslant \frac{4}{5}$. Then it is clear that Kajii and Morris' (1997a) conditions do not apply in such a game. Nonetheless, let us see what are the payoffs when a player believes with probability at least p that the other players will play in the set $\{M, R\}$. Subject to this (set of) beliefs, playing L implies a payoff of at most $5 - 4p$. Moreover, under these beliefs, playing M or R implies, respectively a payoff of at least $4 - p$ and $3.5p$. Therefore, it is easy to show that for any belief of a player that assigns a probability weakly superior (respectively strictly superior) to $\frac{1}{3}$ to the other player playing in $\{M, R\}$, he has a best reply in $\{M, R\}$ (respectively all his best replies are in $\{M, R\}$). Therefore $\{M, R\} \times \{M, R\}$ is a (p, p)-best response set (respectively strict (p, p)-best response set) for any $p \geqslant \frac{1}{3}$. (Note that $\{M, R\} \times \{M, R\}$ does not contain any proper subset that is a $(\frac{1}{3}, \frac{1}{3})$-best response set, therefore it is a minimal (strict) $(\frac{1}{3}, \frac{1}{3})$-best response set.)

[7] Compact-continuous games are games where for each player, the set of actions is a compact subset of a metric space and payoff functions are continuous. In particular, this endows finite games.

Observe that in the game where the set of actions of each player is restricted to $\{M, R\} \times \{M, R\}$, playing R is strictly dominant for both players. This implies that the action distribution μ such that $\mu(R, R) = 1$ is the only correlated equilibrium with support in $\{M, R\} \times \{M, R\}$.

2.3. *Example 2*

Consider the following 3×3 game. Each player has again three possible actions: L, M or R. Payoffs are given by the following matrix.

Player 2

Player 1		L	M	R
	L	5, 5	1, 4	1, 0
	M	4, 1	3, 4	5, 3
	R	0, 1	4, 3	4, 4

This game has a unique strict Nash equilibrium (L, L) which is (p, p)-dominant for $p \geqslant \frac{4}{5}$. Moreover, there exists a mixed NE μ where $\mu_i(M) = \mu_i(R) = \frac{1}{2}$ for each player i. Again, Kajii and Morris' (1997a) conditions do not apply. Now, if a player believes with probability at least p that the other player will play in the set $\{M, R\}$, playing L implies for both players a payoff of at most $5 - 4p$. Moreover, under these beliefs, playing M or R implies, respectively, a payoff of at least $4 - p$ and $4p$. Therefore, it is easy to show that for any belief of a player that assigns a probability weakly superior (respectively strictly superior) to $\frac{1}{3}$ to the other player playing in $\{M, R\}$, he has a best reply in $\{M, R\}$ (respectively all his best replies are in $\{M, R\}$). Therefore $\{M, R\} \times \{M, R\}$ is a (p, p)-best response set (respectively strict (p, p)-best response set) for any $p \geqslant \frac{1}{3}$. (Notice that $\{M, R\} \times \{M, R\}$ does not contain any proper subset that is a $(\frac{1}{3}, \frac{1}{3})$-best response set, therefore it is a minimal (strict) $(\frac{1}{3}, \frac{1}{3})$-best response set.)

Note that here again, one can show that there exists a unique correlated equilibrium with support in $\{M, R\} \times \{M, R\}$. This correlated equilibrium is the unique mixed Nash equilibrium with support in $\{M, R\} \times \{M, R\}$. The associated distribution of action μ is such that $\mu(R, R) = \mu(M, M) = \mu(M, R) = \mu(R, M) = \frac{1}{4}$.

Our results will prove that the existence of such sets will be an important feature to understand the robustness of equilibria.

3. Robustness

3.1. *Associated incomplete information games*

Let us first define an *information system* as a structure $IS = [\Omega, \Im, \{Q_i\}_{i \in \Im}, P]$ where Ω is a countable state space; $\Im = \{1, \ldots, I\}$ is the collection of players; for each player i, Q_i is a partition of the state space; P is a probability measure on the state space. We write $P(\omega)$ for the probability of the singleton event $\{\omega\}$ and $Q_i(\omega)$ for the (unique) element of Q_i containing ω. Therefore, an *incomplete information game* consists of $U = [IS, \{A_i\}_{i \in \Im}, \{u_i\}_{i \in \Im}]$ where IS is an information system as described previously; for each player i, A_i is his action set; and $u_i : A \times \Omega \to \mathbb{R}$ is a bounded state dependent payoff function. Our way to compare complete and incomplete information games is directly inspired from Kajii and Morris (1997a). We say that an incomplete information game U embeds the complete information game Γ if U satisfies the following conditions: (1) the collection of players $\Im = \{1, \dots, I\}$, and (2) their (pure) action sets $A_1, \dots, A_I$ are the same as in Γ. We denote $E(\Gamma)$ the set of incomplete information games which embed Γ.

Throughout the paper we will restrict our attention to incomplete information games where every information set of every player is possible, that is $P[Q_i(\omega)] > 0$ for all $i \in \Im$ and $\omega \in \Omega$. Then the conditional probability of an event is always well defined by Bayes Rule.

A (mixed) strategy for player i is a Q_i-measurable function $\sigma_i : \Omega \to \Delta(A_i)$. We denote by $\sigma_i(a_i \mid \omega)$ the probability that action a_i is chosen given ω under σ_i. A strategy profile is a function $\sigma = (\sigma_i)_{i\in\Im}$ where σ_i is a strategy for player i. We denote by $\sigma(a \mid \omega)$ the probability that action profile a is chosen given ω under σ; we write σ_{-i} for $(\sigma_j)_{j\neq i}$; when no confusion arises, we extend the domain of each u_i to mixed strategies and thus write $u_i(\sigma(\omega), \omega)$ for $\sum_{a\in A} u_i(a, \omega)\sigma(a \mid \omega)$. Now the payoff of strategy profile σ to player i is given by the expected utility $\sum_{\omega\in\Omega}\sum_{a\in A} u_i(a, \omega)\sigma(a \mid \omega)P(\omega)$ which can be written as $\sum_{\omega\in\Omega} u_i(\sigma(\omega), \omega)P(\omega)$.

Definition 4. A strategy profile σ is a (Bayesian) Nash equilibrium of U if, for each $i \in \Im$, $a_i \in A_i$, and $\omega \in \Omega$,

$$\sum_{\omega'\in Q_i(\omega)} u_i\big(\sigma(\omega'), \omega'\big)P\big[\omega' \mid Q_i(\omega)\big] \geqslant \sum_{\omega'\in Q_i(\omega)} u_i\big(a_i, \sigma_{-i}(\omega'), \omega'\big)P\big[\omega' \mid Q_i(\omega)\big].$$

Let $\sigma_P \in \Delta(A)$ be such that $\sigma_P(a) = \sum_{\omega\in\Omega} \sigma(a \mid \omega)P(\omega)$. In the sequel, we call σ_P an action distribution generated by σ.

Kajii and Morris (1997a) formalized the idea that an incomplete information game U is close to a complete information game Γ if with high probability, the payoff structure under U is equal to that under Γ and every player knows his payoff. Thus, for each incomplete information game $U \in E(\Gamma)$, write Ω_U for the set of states where payoffs are given by Γ, and every player knows his payoff:

$$\Omega_U \equiv \big\{\omega \in \Omega \mid u_i(a, \omega') = g_i(a) \text{ for all } a \in A,\ \omega' \in Q_i(\omega),\ \text{and } i \in \Im\big\}.$$

Definition 5. The incomplete information game U is an ε-elaboration of Γ if $U \in E(\Gamma)$ and $P[\Omega_U] = 1 - \varepsilon$. Let $E(\Gamma, \varepsilon)$ be the set of all ε-elaborations of Γ.

Definition 6. An action distribution $\mu \in \Delta(A)$ is robust to incomplete information in Γ if, for every $\delta > 0$, there exists $\bar\varepsilon > 0$ such that, for all $0 \leqslant \varepsilon \leqslant \bar\varepsilon$, every $U \in E(\Gamma, \varepsilon)$ has a Bayesian Nash equilibrium σ such that[8] $\max_{a\in A}|\mu(a) - \sigma_P(a)| \leqslant \delta$.

Remark 4. $\mu \in \Delta(A)$ is not robust if there exists $\delta > 0$ and a sequence of incomplete information games $U^k \in E[\Gamma, \varepsilon^k]$ with $\varepsilon^k \to 0$ such that every Bayesian Nash equilibrium σ^k of U^k with induced action distribution σ_P^k is such that $\max_{a\in A} |\mu(a) - \sigma_P^k(a)| > \delta$. In that sense, μ is not a robust prediction.

We refer the reader to Kajii and Morris (1997a, 1997b) for a discussion of the strength of this test and alternative routes to weaken it.

[8] As in Kajii and Morris (1997a), we measure distance between action distributions by the max norm.

3.2. Belief operator and the critical path result

In this section, we introduce belief operators and the notion of common **p**-belief developed by Monderer and Samet (1989) and then we report Kajii and Morris' (1997a) *Critical Path Result*. Given an information structure $[\Omega, \Im, \{Q_i\}_{i\in\Im}, P]$, for any number $p_i \in (0, 1]$, define for each $E \subseteq \Omega$ $B_i^{p_i}(E) \equiv \{\omega \in \Omega\colon\ P[E \mid Q_i(\omega)] \geqslant p_i\}$.

That is, $B_i^{p_i}(E)$ is the set of states where player i believes E with probability at least p_i. For any row vector $\mathbf{p} = (p_1, \ldots, p_I) \in (0, 1]^I$, $B_*^{\mathbf{p}}(E) \equiv \bigcap_{i\in\Im} B_i^{p_i}(E)$; $B_*^{\mathbf{p}}(E)$ is the set of states where E is **p**-believed, i.e., each player i believes E with probability at least p_i. An event is **p**-evident if it is **p**-believed whenever it is true, i.e. $E \subseteq B_*^{\mathbf{p}}(E)$. An event is common **p**-belief if it is **p**-believed, it is **p**-believed that it is **p**-believed, etc.; thus E is common **p**-believed at ω if $\omega \in C^{\mathbf{p}}(E) \equiv \bigcap_{n\geqslant 1}[B_*^{\mathbf{p}}]^n(E)$.

Kajii and Morris (1997a, Corollary 4-3, p. 1296) showed a connection between the ex ante probability of an event E and the ex ante probability of the event $C^{\mathbf{p}}(E)$ when $\sum_{i\in\Im} p_i < 1$. Their result shows that if $\sum_{i\in\Im} p_i < 1$, then $P[C^{\mathbf{p}}(E)]$ is close to 1 whenever $P[E]$ is close to 1, regardless of the state space. Their proposition[9] is the following:

Proposition 1 *((The Critical Path Result)). If $\sum_{i\in\Im} p_i < 1$, then in any information system $[\Omega, \Im, \{Q_i\}_{i\in\Im}, P]$, all events E satisfy*:

$$P\big[C^{\mathbf{p}}(E)\big] \geqslant 1 - \big(1 - P[E]\big)\left(1 + \sum_{i\in\Im} \frac{p_i}{1 - p_i}\right)\left(\frac{1 - \min_{i\in\Im}(p_i)}{1 - \sum_{i\in\Im} p_i}\right).$$

4. Main results

In this section we will state and prove our main results. We first build on the concept of **p**-best response set in order to provide a sufficient condition for the robustness of an equilibrium in Γ. Then we consider the strict **p**-best response set that allows us to provide a close sufficient condition under which Γ has a unique robust equilibrium.

Theorem 1. *Let S be a* **p***-best response set with $\sum_{i\in\Im} p_i < 1$ of Γ. If there exists a unique correlated equilibrium μ^* with $Supp(\mu^*) \subseteq S$ then μ^* is a robust equilibrium of Γ.*

The statement of the theorem implies that such a μ^* is a Nash equilibrium. Let us see why it is so. First take the modified game where for each player i the action set is restricted to S_i. In this modified game, μ^* is the unique correlated equilibrium and hence it is a Nash equilibrium. Thus in the original game, under μ^*, no player has an incentive to deviate from his strategy in μ^* to an action in S_i. But under μ^*, each player i believes with probability one that the other players will play in S_{-i}. So, by definition of a **p**-best response set, player i has a best reply in S_i and so does not have any incentive to deviate from his strategy in μ^* to an action outside S_i. Therefore μ^* is a Nash equilibrium of the original game. We now move to the proof of Theorem 1. In order to do so, we must first prove some useful lemmas.

[9] Kajii and Morris (1997a) used in the proof of their robustness result a slightly modified version of *The Critical Path Result*. Their version relies on a particular class of events. However, we use their Corollary 4-3 [p. 1296] which applies to any event.

Lemma 1. *Let $S \equiv \times_{i\in\Im} S_i$ be a* **p***-best response set in Γ. Consider any $U \in E(\Gamma)$, and let $F \subseteq \Omega_U$ be a* **p***-evident event. Then U has a Bayesian Nash equilibrium σ such that $\sum_{a_i\in S_i} \sigma_i(a_i \mid \omega) = 1$ for all $i \in \Im$ and $\omega \in F$.*

Proof. Let $F_i = B_i^{p_i}(F)$, so $F \subseteq \bigcap_{i\in\Im} F_i$. Note that for all $\omega' \in Q_i(\omega)$, $P[F \mid Q_i(\omega)] = P[F \mid Q_i(\omega')]$, thus we have that $F_i = \bigcup_{\omega'\in F_i} Q_i(\omega')$, otherwise stated, F_i is a Q_i-measurable set. Consider the modified incomplete information game $U' = [IS, \{A_i\}_{i\in\Im}, \{u_i'\}_{i\in\Im}]$ where we modify payoffs of each player $i \in \Im$ (only at states that belong to F_i) involving strategies outside S_i such that these strategies become strictly dominated (i.e. together with the fact that F_i is a Q_i-measurable set, it is sufficient to assume that for all $i \in \Im$, for all $\omega' \in F_i$, for all $\bar{a}_i \notin S_i$, there exists $a_i \in S_i$ such that $u_i'(a_i, a_{-i}, \omega') = u_i(a_i, a_{-i}, \omega') > u_i'(\bar{a}_i, a_{-i}, \omega')$ for any $a_{-i} \in A_{-i}$). There exists a Bayesian Nash equilibrium σ of the modified game U' where, by construction, each player i's strategy satisfies $\sum_{a_i\in S_i} \sigma_i(a_i \mid \omega') = 1$ for all $\omega' \in F_i$ and $i \in \Im$. We shall show that σ is an equilibrium of U. Let $\omega \notin F_i$. As noted earlier, $F_i = \bigcup_{\omega'\in F_i} Q_i(\omega')$ then (recall that Q_i is a partition of Ω) $Q_i(\omega) \cap F_i = \emptyset$. Thus it should be clear that for all $i \in \Im$, σ_i is a best response to σ_{-i} at ω (since i's payoffs from U to U' change only at F_i). Now let $\omega \in F_i$. Then $P[F \mid Q_i(\omega)] \geqslant p_i$; thus under σ, the conditional probability that player i assigns to the other players playing in S_{-i} is weakly above p_i. Since $F \subseteq \Omega_U$ and $Q_i(\omega) \cap F \neq \emptyset$, by definition of Ω_U, payoffs are given by Γ at $\omega \in F_i$. Moreover, since F_i is a Q_i-measurable set, i knows that his payoffs are given by Γ. Because S is a **p**-best response set in Γ: there exists $\hat{a}_i \in S_i$ such that

$$\sum_{\omega'\in Q_i(\omega)} u_i\big(\hat{a}_i, \sigma_{-i}(\omega'), \omega'\big) P\big[\omega' \mid Q_i(\omega)\big] \geqslant \sum_{\omega'\in Q_i(\omega)} u_i\big(a_i, \sigma_{-i}(\omega'), \omega'\big) P\big[\omega' \mid Q_i(\omega)\big]$$

for all $a_i \notin S_i$. But

$$\sum_{\omega'\in Q_i(\omega)} u_i\big(\sigma(\omega'), \omega'\big) P\big[\omega' \mid Q_i(\omega)\big] \geqslant \sum_{\omega'\in Q_i(\omega)} u_i\big(\hat{a}_i, \sigma_{-i}(\omega'), \omega'\big) P\big[\omega' \mid Q_i(\omega)\big].$$

Then at $\omega \in F_i$, σ_i is weakly better than any $a_i \notin S_i$. But it is also weakly better than any $a_i \in S_i$ (since it is an equilibrium of U'). Thus σ is also a Bayesian Nash equilibrium of the original game U. □

As noted in Kajii and Morris (1997a) (assuming $\mathbf{p} > 0$), $C^{\mathbf{p}}(\Omega_U)$ is the largest **p**-evident event set contained in Ω_U. Thus, an immediate corollary can be stated:

Corollary 1. *Let S be a* **p***-best response set of Γ. Consider any $U \in E(\Gamma)$, then U has a Bayesian Nash equilibrium σ such that $\sum_{a_i\in S_i} \sigma_i(a_i \mid \omega) = 1$ for all $i \in \Im$ and $\omega \in C^{\mathbf{p}}(\Omega_U)$.*

We move to the following important lemma in the proof of Theorem 1.

Lemma 2. *Let S be a* **p***-best response set of Γ with $\sum_{i\in\Im} p_i < 1$. Then for any sequence of ε^k-elaboration U^k of Γ, where $\varepsilon^k \to 0$ (as $k \to +\infty$), every U^k has a Bayesian Nash equilibrium σ^k with induced action distribution σ_P^k such that $\sum_{a\in S} \sigma_P^k(a) \to 1$ (as $k \to +\infty$).*

Proof. By construction, for any U^k, we have $P[\Omega_{U^k}] = 1 - \varepsilon^k$. *The Critical Path Result* (Proposition 1) implies $P[C^{\mathbf{p}}(\Omega_{U^k})] \geqslant 1 - \delta^k$ where $\delta^k = \varepsilon^k \times (1 + \sum_{i\in\Im} \frac{p_i}{1-p_i}) \frac{1-\min_{i\in\Im}(p_i)}{1-\sum_{i\in\Im} p_i}$. Clearly, $\delta^k \to 0$ (as $k \to +\infty$). By Corollary 1, there exists a Bayesian Nash equilibrium σ^k of U^k with $\sum_{a_i\in S_i} \sigma_i^k(a_i \mid \omega) = 1$ for all $i \in \Im$, for all $\omega \in C^{\mathbf{p}}(\Omega_{U^k})$. This completes the proof. □

Lemma 3. *Suppose $\varepsilon^k \to 0$ as $k \to +\infty$. Let σ^k be a Bayesian Nash equilibrium of $U^k \in E[\Gamma, \varepsilon^k]$ and let σ_P^k be the action distribution generated by σ^k. Then σ_P^k has a subsequence which converges to some correlated equilibrium of Γ.*

Proof. See Kajii and Morris (1997a, Corollary 3-5, p. 1294). $\square$

Proof of Theorem 1. Now the proof of Theorem 1 is completed as follows.

Suppose that μ^* is not robust. Then, there exists $\delta > 0$ and a sequence of $U^k \in E(\Gamma, \varepsilon^k)$ where $\varepsilon^k \to 0$ (as $k \to +\infty$) such that for every equilibrium σ^k of U^k with induced action distribution σ_P^k, $\max_{a\in A} |\mu^*(a) - \sigma_P^k(a)| > \delta$ for all k. Together with Lemma 2, we can restate this by: there exists $\delta > 0$ and a sequence of $U^k \in E(\Gamma, \varepsilon^k)$ where $\varepsilon^k \to 0$ (as $k \to +\infty$) which has a Bayesian Nash equilibrium $\tilde{\sigma}^k$ with induced action distribution $\tilde{\sigma}_P^k$ satisfying $\sum_{a\in S} \tilde{\sigma}_P^k(a) \to 1$ (as $k \to +\infty$) and $\max_{a\in A} |\mu^*(a) - \tilde{\sigma}_P^k(a)| > \delta$ for all k. But by Lemma 3, $\{\tilde{\sigma}_P^k\}_{k=1}^{\infty}$ has a subsequence which converges to some correlated equilibrium $\bar{\mu}$ of Γ which must be different from μ^* since $\max_{a\in A} |\mu^*(a) - \tilde{\sigma}_P^k(a)| > \delta$ for all k. Since μ^* is the unique CE with support in S, we have $Supp(\bar{\mu}) \nsubseteq S$. This contradicts $\sum_{a\in S} \tilde{\sigma}_P^k(a) \to 1$ (as $k \to +\infty$). Therefore we have shown that μ^* is robust. $\square$

Our condition generalizes Kajii and Morris (1997a) since a unique correlated equilibrium implies that there exists a trivial set (the set of all available action profiles) which is a **p**-best response (with $\sum_{i\in\Im} p_i < 1$) and which has a unique correlated equilibrium whose support is in that set. The second condition for robustness in Kajii and Morris (1997a) states that if an equilibrium a^* is a **p**-dominant equilibrium with $\sum_{i\in\Im} p_i < 1$ then it is a robust equilibrium. Clearly at such an equilibrium the singleton set $\{a^*\}$ is a **p**-best response set with $\sum_{i\in\Im} p_i < 1$ and contains a unique correlated equilibrium whose support is trivially in $\{a^*\}$. In the sequel, we provide a slight refinement of the condition of Theorem 1, under which Γ has a unique robust equilibrium.

Theorem 2. *Let S be a strict **p**-best response set with $\sum_{i\in\Im} p_i < 1$ of Γ. If there exists a unique correlated equilibrium μ^* with $Supp(\mu^*) \subseteq S$ then μ^* is the unique robust equilibrium of Γ.*

Proof. Since S is a strict **p**-best response set, it is a **p**-best response set. Thus, μ^* meets the conditions of Theorem 1 and is therefore a robust equilibrium. Let us show that it is the unique robust equilibrium of Γ. In order to do so, we build on a Kajii and Morris' (1997a) proof. For our purpose, it is sufficient to show that for all $\varepsilon > 0$, there exists $U \in E(\Gamma, \varepsilon)$ such that the set of Bayesian NE of U consists in playing in S.

Let $q_i = (p_i / \sum_{j\in\Im} p_j) > p_i$ for each $i \in \Im$. It follows that $\sum_{i\in\Im} q_i = 1$. Now fix $\varepsilon > 0$ and let $\Omega = \Im \times \mathbb{N}_+$ and $P(i,k) = \varepsilon(1-\varepsilon)^k q_i$. Let each Q_i consist of (i) the event $E_i^0 = \{(j,0)_{j\neq i}\}$; and (ii) all events of the form $E_i^k = \{(i,k-1), (j,k)_{j\neq i}\}$, for each integer $k \geqslant 1$. Let

$$u_i(a,\omega) = \begin{cases} g_i(a) & \text{if } \omega \notin E_i^0, \\ 1 & \text{if } \omega \in E_i^0 \text{ and } a_i \in S_i, \\ 0 & \text{if } \omega \in E_i^0 \text{ and } a_i \notin S_i. \end{cases}$$

And let σ^* be any NE of $U \in E(\Gamma, \varepsilon)$. We show that for all $i \in \Im$, $\omega \in \Omega$, $\sum_{a_i\in S_i} \sigma_i^*(a_i \mid \omega) = 1$. By construction, $\sum_{a_i\in S_i} \sigma_i^*(a_i \mid \omega) = 1$ for all $\omega \in E_i^0$ and $i \in \Im$. Now our inductive hypothesis is that $\sum_{a_i \in S_i} \sigma_i^*(a_i \mid \omega) = 1$ for all $\omega \in E_i^k$ and $i \in \Im$. Consider any $\omega \in E_i^{k+1}$, by construction of the state space, we have

$$P\big[\{(i,k)\} \mid Q_i(\omega)\big] = \frac{\varepsilon(1-\varepsilon)^k q_i}{\varepsilon(1-\varepsilon)^k q_i + \sum_{j \neq i} \varepsilon(1-\varepsilon)^{k+1} q_j} = \frac{q_i}{q_i + (1-\varepsilon)\sum_{j \neq i} q_j} > q_i > p_i.$$

Thus by the inductive hypothesis, each agent i assigns a probability strictly above p_i to the other players playing in S_{-i}. But as S is a strict **p**-best response set, this implies that $\sum_{a_i \in S_i} \sigma_i^*(a_i \mid \omega) = 1$ for all $\omega \in E_i^{k+1}$ and $i \in \Im$. Thus our inductive hypothesis holds for $k+1$. $\square$

Theorem 2 is a generalization of another result of Kajii and Morris (1997a) that states that if an equilibrium a^* is a strict **p**-dominant equilibrium with $\sum_{i \in \Im} p_i < 1$ then it is the unique robust equilibrium. Clearly at such an equilibrium the singleton set $\{a^*\}$ is a strict **p**-best response set with $\sum_{i \in \Im} p_i < 1$ and contains a unique correlated equilibrium whose support is trivially in $\{a^*\}$.

The two simple examples in Section 2 show that our condition applies when those of Kajii and Morris (1997a) do not, thus showing that our condition is strictly weaker. In Example 1, it is easy to see that none of the sufficient conditions of Kajii and Morris (1997a) are satisfied. Nonetheless, as shown above, $\{M,R\} \times \{M,R\}$ is a (strict) $(\frac{1}{3},\frac{1}{3})$-best response set and the action distribution μ such that $\mu(R,R) = 1$ is the only correlated equilibrium whose support is in $\{M,R\} \times \{M,R\}$. Hence, it is the unique robust equilibrium. The second example has a unique strict Nash equilibrium (L,L). Recall that we have shown that $\{M,R\} \times \{M,R\}$ is a (strict) $(\frac{1}{3},\frac{1}{3})$-best response set. The unique mixed Nash equilibrium μ with support in $\{M,R\} \times \{M,R\}$, which is such that $\mu_i(M) = \mu_i(R) = \frac{1}{2}$ (for both i) defines a distribution of actions that is the unique correlated equilibrium with support in $\{M,R\} \times \{M,R\}$. Thus this game satisfies our conditions and μ is robust. It is the unique robust equilibrium and therefore, the unique strict Nash equilibrium is not robust.

5. Related literature: potential techniques

A recent work of Morris and Ui (2005) has introduced a concept of Local Potential function.[10] In this section, we show that our theorem can be seen as a particular case of Morris and Ui's (2005) work. Nevertheless finding Local Potential functions can be a rather hard task. Our condition is much easier to manipulate and seems to be a natural first step to see if Morris and Ui's condition applies.

Let P_i be a partition of A_i such that P_i is linearly ordered by the order relation $\leqslant_i$ for $i \in \Im$. Let $\underline{Z}_i$ and $\overline{Z}_i$ be the smallest and the largest elements of P_i, respectively. The corresponding product order relations over P and A are denoted by $\leqslant_\Im$, and those over P_{-i} and A_{-i} are denoted by $\leqslant_{-i}$, respectively. We will also note $P_i(a_i)$ for the (unique) element of P_i containing a_i.

Let $Z_i^+ \in P_i$ be the smallest element in P_i larger than $Z_i \neq \overline{Z}_i$ and $Z_i^- \in P_i$ be the largest element in P_i smaller than $Z_i \neq \underline{Z}_i$. We first state some definitions.

[10] In fact, Morris and Ui (2005) has introduced Generalized Potential functions. We will restrict attention to Local Potential functions that are a particular form of these functions.

Definition 7. A complete information game Γ satisfies diminishing marginal returns if, for each $i \in \Im$ and $a_{-i} \in A_{-i}$,

$$g_i\big(a_i^+, a_{-i}\big) - g_i(a_i, a_{-i}) \leqslant g_i(a_i, a_{-i}) - g_i(a_i^-, a_{-i})$$

for $a_i \notin \underline{Z}_i \cup \overline{Z}_i$, $a_i^+ \in P_i(a_i)^+$, and $a_i^- \in P_i(a_i)^-$.

Definition 8. Let $X^* \in P$ be given. A P-measurable function $v : A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a^* \in X^*$ and $a \notin X^*$ is a local potential function of Γ if, for each $i \in \Im$, $Z_i >_i X_i^*$,

$$\max_{a_i' \in Z_i^-} \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i\big(a_i', a_{-i}\big) \geqslant \max_{a_i' \in Z_i} \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i\big(a_i', a_{-i}\big)$$

for all $\lambda \in \Delta(A_{-i})$ such that

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) v(a_i^-, a_{-i}) \geqslant \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) v(a_i, a_{-i}),$$

where $a_i^- \in Z_i^-$ and $a_i \in Z_i$; and symmetrically, for each $i \in \Im$, $Z_i <_i X_i^*$,

$$\max_{a_i' \in Z_i^+} \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i\big(a_i', a_{-i}\big) \geqslant \max_{a_i' \in Z_i} \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i\big(a_i', a_{-i}\big)$$

for all $\lambda \in \Delta(A_{-i})$ such that

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) v\big(a_i^+, a_{-i}\big) \geqslant \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) v(a_i, a_{-i}),$$

where $a_i^+ \in Z_i^+$ and $a_i \in Z_i$. A partition element $X^* \in P$ is called a *local potential maximizer* (*LP*-maximizer).

Definition 9. A complete information game Γ satisfies strategic complementarities if, for each $i \in \Im$,

$$g_i(a_i, a_{-i}) - g_i\big(a_i', a_{-i}\big) \geqslant g_i\big(a_i, a_{-i}'\big) - g_i\big(a_i', a_{-i}'\big)$$

for all $a_i, a_i' \in A_i$ and $a_{-i}, a_{-i}' \in A_{-i}$ such that $P_i(a_i) >_i P_i(a_i')$ and $P_{-i}(a_{-i}) >_{-i} P_{-i}(a_{-i}')$. A function $v : A \to \mathbb{R}$ satisfies strategic complementarities if an identical interest game Γ with $g_i = v$ for all $i \in \Im$ satisfies strategic complementarities.

In the particular case where for each player i, $P_i = \{S_i, A_i \backslash S_i\}$, we have the following characterization:

Lemma 4. *Let $P_i = \{S_i, A_i \backslash S_i\}$ with $S_i <_i A_i \backslash S_i$ for all $i \in \Im$. $v : A \to \mathbb{R}$ is a local potential function with an LP-maximizer $S = \times_{i \in \Im} S_i$ if and only if*

(i) *$v(a) > v(a')$ for all $a \in S$ and $a' \notin S$,*
(ii) *v is a P-measurable function,*
(iii) *for all $i \in \Im$, if for $a_i^- \in S_i$, and $a_i \in A_i \backslash S_i$, $\lambda \in \Delta(A_{-i})$ satisfies*

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) v(a_i^-, a_{-i}) \geqslant \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) v(a_i, a_{-i}),$$

then for some $a_i \in S_i$ *and all* $a_i' \in A_i$,

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i(a_i, a_{-i}) \geqslant \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i\bigl(a_i', a_{-i}\bigr).$$

Proof. The straightforward proof is omitted. □

The following proposition links our approach with that of Morris and Ui (2005).

Proposition 2. *If* Γ *has a* **p**-*best response set* S *with* $\sum_{i \in \Im} p_i < 1$, *then* Γ *has a local potential function* $v : A \to \mathbb{R}$ *with an LP-maximizer* S *such that*:

$$v(a) = \begin{cases} 1 - \sum_{i \in \Im} p_i & \text{if } a \in S, \\ -\sum_{i \in N} p_i & \text{if } a_i \in S_i \text{ for } i \in N \text{ and } a_i \notin S_i \text{ for } i \notin N. \end{cases}$$

In addition, v *satisifes strategic complementarity.*

Proof. Note that

$$v(a_i, a_{-i}) - v\bigl(a_i', a_{-i}\bigr) = \begin{cases} 1 - p_i & \text{if } a_{-i} \in S_{-i}, \\ -p_i & \text{otherwise}, \end{cases}$$

where $a_i \in S_i$ and $a_i' \notin S_i$. Thus, v satisfies strategic complementarities and (i) and (ii) are satisfied (recall that $\sum_{i \in \Im} p_i < 1$). Let us prove that (iii) is also satisfied.

Suppose that for $a_i^- \in S_i$, and $a_i \in A_i \backslash S_i$, $\lambda \in \Delta(A_{-i})$ satisfies

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) v(a_i^-, a_{-i}) \geqslant \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) v(a_i, a_{-i})$$

then,

$$\begin{aligned} &\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) \bigl[v(a_i^-, a_{-i}) - v(a_i, a_{-i})\bigr] \\ &\quad = \sum_{a_{-i} \in S_{-i}} \lambda(a_{-i})(1 - p_i) + \sum_{a_{-i} \notin S_{-i}} \lambda(a_{-i})(-p_i) = \sum_{a_{-i} \in S_{-i}} \lambda(a_{-i}) - p_i \geqslant 0. \end{aligned}$$

Thus $\lambda \in \Pi_w^{p_i}(S_{-i})$. Since S is a **p**-best response set, there exists $a_i \in S_i$ such that

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i(a_i, a_{-i}) \geqslant \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i\bigl(a_i', a_{-i}\bigr)$$

for all $a_i' \in A_i$. Thus (iii) is satisfied and the proof is completed. □

Note also that Γ satifies diminishing marginal returns in the trivial sense. We now restate Proposition 3 in Morris and Ui (2005) that implies our main theorem.[11]

Proposition 3. *Suppose that* Γ *has a local potential function* $v : A \to \mathbb{R}$ *with an LP-maximizer* X^*. *Assume that* Γ *admits a unique correlated equilibrium* μ^* *with* $Supp(\mu^*) \subseteq X^*$. *If* Γ *satisfies diminishing marginal returns, and if* Γ *or* v *satisfies strategic complementarities, then* μ^* *is a robust equilibrium of* Γ.

[11] Note that the original proposition of Morris and Ui (2005) relies on robustness of set of correlated equilibria.

In the spirit of Morris and Ui's (2005) work, it shows that higher order beliefs techniques and potential methods are closely related.

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