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Analytical solutions to a structural signal extraction model: Lucas 1972 revisited☆

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Abstract

I construct in this article a simple adaptation of Lucas' 1972 article on the neutrality of money. A closed-form solution is derived, which allows to understand a number of positive and normative properties of the model. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

A most interesting contribution of Lucas's 1972 "Neutrality of money" article is the issue of imperfect "signal extraction": Because of incomplete information, agents cannot disentangle monetary and real shocks, and as a result money, which would be neutral under full information, has an impact on labor supplies

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and output. Very strangely, and although this 1972 article is extremely often cited, there seems to have been to this date very few other fully rigorous macroeconomic models imbedding this signal extraction problem.[1] In fact most of the literature, as well as textbook presentations, is based on various adaptations of another model developed in Lucas (1973). This article presents a suggestive and simple version of the signal extraction problem and the real-monetary confusion with a closed form solution, but, unlike the 1972 model, it does not have explicit microfoundations. In times where the emphasis is more and more on "structural" macroeconomic models with explicit foundations, this is not an optimal state of affairs.

So the purpose of this article is to construct a simple, but general enough, variant of Lucas's 1972 model, augmented with aggregate supply shocks as in Wallace (1992), for which we shall be able to derive closed form solutions. This will allow to understand better a number of positive and normative properties of the model, which seems a useful first step towards progress in this important domain.[2]

2. The model

The economy consists of many isolated subeconomies, "islands", indexed by $j \in \{1, \dots, J\}$, where J is a very large number. Each island j operates in a decentralized manner, but islands are hit by various shocks, some of which are correlated across islands. We shall now describe a representative island, whose index j will be omitted in this and the next section to simplify notation.

Each island is a "Samuelsonian" (Samuelson (1958)) overlapping generations economy. Generation t has N_t members, where N_t is an i.i.d. stochastic variable. Households of generation t work L_t when young (i.e. in period t) and consume C_{t+1} when old (i.e. in period $t+1$). The utility function of generation t households is

$$U_t = \frac{C_{t+1}^{\alpha} - 1}{\alpha} - \frac{L_t^{\beta}}{\beta} \tag{1}$$

[1] A notable exception is the earlier article by Hahm (1987), who carries calculations very similar to those of this article for the case without aggregate shocks. He further develops an insightful analysis of endogenous information acquisition in this model, which should be of interest to all readers concerned with the issue of imperfect signal extraction. I wish to thank Robert King for making me aware of this article.

[2] Earlier articles with similar purposes include Azariadis (1981), who derives a solution of the Lucas model for a particular utility function (which actually corresponds in our parameterization below to $\alpha = 1$ and $\beta = 2$) and McCallum (1984), who develops a linearized version.

with $\alpha \leq 1$ and $\beta \geq 1$. The limit case $\alpha = 0$ corresponds to a logarithmic utility function. It is most often assumed that consumption and leisure are gross substitutes, which corresponds to $\alpha > 0$.

The technology is very simple: one unit of labor produces one unit of perishable output. As a result the resource constraint is in period t:

$$N_{t-1} C_t = N_t L_t. \tag{2}$$

Of course, since households consume one period after they have worked, they must transfer wealth from one period to the next, which is done by accumulating money, the only store of value. Call M_t the quantity of money available in period t. All individual money holdings are multiplied between periods $t-1$ and t by a coefficient X_t common to all agents,[3] so that total money supply evolves according to:

$$M_t = X_t M_{t-1}. \tag{3}$$

Denote by P_t the price of the good, which is also the money wage because of the simple linear technology. Old agents have a quantity of money M_t which they spend entirely. If young agents supply a quantity of labor L_t each, the condition of equilibrium on the goods market is simply

$$N_t L_t = \frac{M_t}{P_t} = \frac{X_t M_{t-1}}{P_t}. \tag{4}$$

At this stage we need to be more precise about the information available to the young household, as this is an essential element in the signal extraction problem. As in Lucas (1972), we assume that individual agents receive exact information about the magnitudes of N_t and X_t only in period $t+1$. In period t it is assumed that each young agent knows M_{t-1}, and observes only the price level P_t and the ratio $Z_t = X_t / N_t$.

In order to have explicit solutions, we shall finally make particular assumptions about the distributions of the stochastic variables N_t and X_t. Calling n_t and x_t their logarithms, we shall assume that these are independent normal variables with mean zero and variances σ_n^2 and σ_x^2, respectively. The parameter σ_n^2 belongs to the basic "fundamentals" of the economy, whereas σ_x^2 is a policy parameter that can be chosen by the government.

3. Market equilibrium on an island

We shall now study market equilibrium for a particular island j, whose index we continue to omit. Looking at Eqs. (2) and(4), we see that the equilibrium will

[3] This "proportional" money creation is the actual assumption in Lucas (1972), and moreover allows a closed form solution for all feasible values of α and β. The more traditional "lump-sum" monetary creation will be studied in Sections 7 and 8 below.

be fully defined if we can describe the labor supply behavior of young agents. The representative agent will choose L_t so as to maximize his expected utility, i.e. he will solve the following program:

$$\text{Maximize } \mathrm{E}_t\left(\frac{C_{t+1}^{\alpha}}{\alpha}\right) - \frac{L_t^{\beta}}{\beta} \quad \text{s.t.} \quad C_{t+1} = \frac{P_t X_{t+1} L_t}{P_{t+1}}. \tag{5}$$

In order to compute the expectation in (5) the agent forms a "theory" about price formation. More precisely the agent will assume that the price P_t and the observed variable Z_t are related in each period by the following relation:

$$P_t = \xi M_{t-1} Z_t^{\gamma}, \tag{6}$$

where ξ and γ are for the moment taken as parametric. Both values will be deduced below from the assumption of rational expectations. Using (6) the expectation term in program (5) can be written as

$$\mathrm{E}_t\left(\frac{C_{t+1}^{\alpha}}{\alpha}\right) = Z_t^{\alpha\gamma} L_t^{\alpha} E_t\left[\frac{1}{\alpha}\left(\frac{X_{t+1}^{1-\gamma} N_{t+1}^{\gamma}}{X_t}\right)^{\alpha}\right]. \tag{7}$$

To compute the expectation in (7), we first note that X_t and $Z_t = X_t/N_t$ are jointly lognormal, so that, conditionally on $z_t = \operatorname{Log} Z_t$, the variable $x_t = \operatorname{Log} X_t$ is normal with mean $\mathrm{E}(x_t \mid z_t) = \rho z_t$ and variance $\operatorname{Var}(x_t \mid z_t) = \rho\sigma_n^2$, where $\rho = \sigma_x^2/(\sigma_x^2 + \sigma_n^2)$.

Secondly we know that, if the logarithm of a stochastic variable is normal with mean μ and variance σ^2, then its expected value is equal to $\exp(\mu + \sigma^2/2)$. Applying this formula to the various terms in (7), we obtain

$$\mathrm{E}_t\left(\frac{C_{t+1}^{\alpha}}{\alpha}\right) = \frac{\Lambda L_t^{\alpha} Z_t^{\alpha(\gamma-\rho)}}{\alpha}, \tag{8}$$

where $\Lambda = \exp\{\alpha^2[\gamma^2\sigma_n^2 + (1-\gamma)^2\sigma_x^2 + \rho\sigma_n^2]/2\}$. Now insert the value obtained in (8) into program (5). We find that the labor decision of young households will be

$$L_t = \Lambda^{1/\beta-\alpha} Z_t^{\alpha(\gamma-\rho)/(\beta-\alpha)}. \tag{9}$$

The last step of the resolution is to determine the values of the parameters γ and ξ which will make the "price theory" (6) a rational one. For that we insert the value of labor supply given by Eq. (9) into the equilibrium Eq. (4). We obtain

$$P_t = \Lambda^{-1/\beta-\alpha} M_{t-1} Z_t^{1-\alpha(\gamma-\rho)/(\beta-\alpha)}. \tag{10}$$

Identification of the two price equations (6) and (10) yields immediately $\xi = \Lambda^{-1/\beta-\alpha}$ and $\gamma = 1 - \alpha(1-\rho)/\beta$. Inserting these values into (9) and (2), and reintroducing the island's index j, we finally obtain the amounts of labor and consumption in period t:

$$L_{jt} = \Lambda^{1/\beta-\alpha}\left(\frac{X_t}{N_{jt}}\right)^{1-\gamma}, \qquad C_{jt} = \Lambda^{1/\beta-\alpha}\frac{X_t^{1-\gamma}N_{jt}^{\gamma}}{N_{jt-1}}. \tag{11}$$

4. The global economy

We can now move to the study of the whole economy by aggregating our results for individual islands. We must now make fully clear how all shocks are related across islands. To be precise, we assume that the monetary shock X_t is common to all islands, whereas the "population shock" N_{jt} has a global and local component. We shall specifically assume that $N_{jt} = N_t\Theta_{jt}$, where the $\theta_{jt} = \operatorname{Log}\Theta_{jt}$ are i.i.d. normal variables with mean zero and variance σ_θ^2.

Clearly, all the analysis of the preceding section will carry through for each single island, simply replacing N_t by $N_{jt} = N_t\Theta_{jt}$. Accordingly formulas (11) still hold, and the final values of the key parameters ρ, γ and Λ are simply obtained replacing σ_n^2 by $\sigma_n^2 + \sigma_\theta^2$:

$$\rho = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_n^2 + \sigma_\theta^2} \qquad \gamma = 1 - \frac{\alpha(1-\rho)}{\beta}, \tag{12}$$

$$\Lambda = \exp\left\{\frac{\alpha^2(\sigma_n^2 + \sigma_\theta^2)}{2}\left[2\rho + (1-\rho)\left(\frac{\beta-\alpha}{\beta}\right)^2\right]\right\}. \tag{13}$$

Now, since we will study below some output–inflation statistics, let us first compute the output and price level on each island:

$$Y_{jt} = \Lambda^{1/\beta-\alpha}N_t^{\gamma}X_t^{1-\gamma}\Theta_{jt}^{\gamma} \qquad P_{jt} = \Lambda^{-1/\beta-\alpha}M_{t-1}\left(\frac{X_t}{N_t\Theta_{jt}}\right)^{\gamma},$$

where M_{t-1} is the quantity of money per island, assumed the same in all islands. Aggregating these across all islands, we obtain average output per island $Y_t = \sum_j Y_{jt}/J$ and the aggregate price level P_t:

$$Y_t = \Lambda^{1/\beta-\alpha}N_t^{\gamma}X_t^{1-\gamma}\frac{1}{J}\sum_j \Theta_{jt}^{\gamma} \qquad P_t = \frac{\sum_j P_{jt}Y_{jt}}{\sum_j Y_{jt}} = \frac{M_{t-1}X_t}{Y_t}.$$

We assume that J is large enough for the law of large numbers to apply for the Θ_{jt}, so that $\sum_j\Theta_{jt}^{\gamma}/J = E(\Theta_{jt}^{\gamma}) = \exp(\gamma^2\sigma_\theta^2/2)$. Define $\Phi = \Lambda^{1/\beta-\alpha}\exp(\gamma^2\sigma_\theta^2/2)$. We obtain the final expressions for aggregate output and price:

$$Y_t = \Phi N_t^{\gamma}X_t^{1-\gamma} \qquad P_t = \frac{M_{t-1}}{\Phi}\left(\frac{X_t}{N_t}\right)^{\gamma}. \tag{14}$$

5. Money, output and inflation

The above explicit solutions allow us to obtain "true" structural relations between the various macroeconomic variables, and to compute relevant macroeconomic correlations. Let us denote by lowercase letters the logarithms of the corresponding uppercase variables. Taking the logarithms of the expressions in (14), we obtain

$$y_t = (1 - \gamma)x_t + \gamma n_t + \phi, \tag{15}$$

$$\pi_t = p_t - p_{t-1} = \gamma x_t + (1 - \gamma)x_{t-1} + \gamma(n_{t-1} - n_t), \tag{16}$$

where $\phi = \operatorname{Log} \Phi$. Eqs. (15) and (16) generalize, by including aggregate supply shocks, similar linear relations obtained in Lucas (1973). A fundamental feature is that the coefficients of these equations depend only on the fundamental data of the model. If $\alpha > 0$, both y_t and π_t are positively correlated with x_t.

5.1. *The output–inflation relation*

The output–inflation tradeoff has been for a long time a hotly debated topic, so an interesting statistics to compute is the covariance between output and inflation. As Wallace (1992) pointed out, this will not necessarily be positive, due to the presence of aggregate supply shocks. We can compute indeed from Eqs. (15) and (16) that $\operatorname{Cov}(y_t,\pi_t) = \gamma(1 - \gamma)\sigma_x^2 - \gamma^2\sigma_n^2$, which, using formulas (12), yields:

$$\operatorname{Cov}(y_t,\pi_t) = \frac{\gamma}{\beta}\left[\frac{\alpha\sigma_x^2\sigma_\theta^2}{\sigma_x^2 + \sigma_n^2 + \sigma_\theta^2} - (\beta - \alpha)\sigma_n^2\right].$$

This formula shows us a number of things: first, if there are no aggregate supply shocks as in Lucas (1972,1973), i.e. if $\sigma_n^2 = 0$ (and $\alpha > 0$, the most usual assumption), then the output–inflation covariance is positive. But the existence of aggregate supply shocks will reduce this covariance, and if all supply shocks are aggregate ones, i.e. if $\sigma_\theta^2 = 0$, then the output–inflation covariance is negative. We see that this formulation is consistent with a large range of observed inflation–output correlations.

5.2. *Money and output*

Our analytical solution also allows to investigate analytically two empirical regularities between money and output studied by Lucas (1973) and Wallace (1992). Actually the variable they consider is not money increases, but increases in nominal output $P_t Y_t/P_{t-1} Y_{t-1}$. However using relations (14) and (3), we find that $P_t Y_t/P_{t-1} Y_{t-1} = X_t$. In this model increases in nominal output are the same as increases in money, so that we shall study the relation between output and money increases, i.e. in logarithms between $x_t = \text{Log}\, X_t$ and $y_t = \text{Log}\, Y_t$.

The first empirical regularity they noted is that there seems to be no tendency in cross country observations for the variance of y_t, σ_y^2, to be proportional to σ_x^2. To see whether this is consistent with our model, we first use Eq. (15) to find $\sigma_y^2 = (1-\gamma)^2\sigma_x^2 + \gamma^2\sigma_n^2$. Using (12) we could express the value of σ_y^2 as a function of the variances of the three shocks. The expression is a bit clumsy, so we use instead the variable $\zeta = \sigma_x^2/(\sigma_\theta^2 + \sigma_n^2)$. For given real shocks, ζ is proportional to σ_x^2. We then find:

$$\sigma_y^2 = \frac{\alpha^2\zeta\sigma_\theta^2}{\beta^2(1+\zeta)^2} + \left[\left(1-\frac{\alpha}{\beta}\right)^2 + \zeta\right]\frac{\sigma_n^2}{1+\zeta}.$$

This formula shows clearly that σ_y^2 is in no way proportional to ζ, and thus neither to σ_x^2.

The second regularity reported by Lucas and Wallace is that, if one regresses y_t on x_t in various countries, the regression coefficient tends to approach zero when σ_x^2 becomes large. To check this, let us go back to Eq. (15) above. The regression coefficient of y_t on x_t will be equal to $1-\gamma$, which, using formulas (12), is equal to

$$1-\gamma = \frac{\alpha}{\beta}(1-\rho) = \frac{\alpha(\sigma_n^2+\sigma_\theta^2)}{\beta(\sigma_x^2+\sigma_n^2+\sigma_\theta^2)}.$$

This regression coefficient is clearly decreasing in the variance of the money shock, and goes to zero when σ_x^2 becomes very large.

6. Welfare analysis

Because the form of monetary policy used so far, i.e. proportional monetary expansions, is a bit unusual, we shall not make a full normative analysis, but rather use our results to throw some light on an intriguing result obtained by Polemarchakis and Weiss (1977). They found indeed, using basically the same parameterization for the utility function as the one above, that for some values of α and β (and specifically $\alpha = 1$ and $\beta = 2$), a "totally random" monetary policy, i.e. here a monetary policy with infinite σ_x^2 corresponding to $\rho = 1$, dominated a deterministic monetary policy.

We shall thus analyze the case where, as in Lucas (1972), the supply shocks are local, i.e. $\sigma_n^2 = 0$. In that case everything happens as if there was a very large constant total population, and individual agents were allocated randomly to islands of various sizes. The probability for an agent to be allocated to an island of size Θ_t (we omit again the island index j to simplify notation) is then equal to the probability that Θ_t occurs multiplied by the "relative size" of the island

$\Theta_t/\mathrm{E}(\Theta_t)$.[4] The expected utility of the representative agent is thus equal to $\mathrm{E}(\Theta_t U_t)/\mathrm{E}(\Theta_t)$.

The analysis carried out in Sections 3 and 4 fully applies in the case of purely "local" shocks, simply replacing N_{jt} by Θ_t. Using (1), (11)–(13), we can compute the expected utility of the representative household $\mathrm{E}(\Theta_t U_t)/\mathrm{E}(\Theta_t)$, and obtain

$$\frac{\mathrm{E}(\Theta_t U_t)}{\mathrm{E}(\Theta_t)} = \frac{1}{\alpha}\exp\left\{\frac{\alpha\sigma_\theta^2}{2}\left[\frac{\alpha^2(3\beta-\alpha)}{\beta(\beta-\alpha)}\rho - \frac{\alpha^2}{\beta} - 2(1-\alpha)\right]\right\} - \frac{1}{\alpha}$$
$$- \frac{1}{\beta}\exp\left\{\frac{\alpha\sigma_\theta^2}{2}\left[\frac{\alpha^2(3\beta-\alpha)}{\beta(\beta-\alpha)}\rho + 2\rho - \frac{\alpha^2}{\beta} - 2(1-\alpha)\right]\right\}. \tag{17}$$

We may note that, since σ_θ^2 is given, we can take ρ as our policy variable instead of σ_x^2 (cf. Eq. (12) with $\sigma_n^2 = 0$). Differentiating (17) with respect to ρ, we find

$$\frac{\partial \mathrm{E}(\Theta_t U_t)}{\partial \rho} = \Omega\left\{\frac{\alpha^2(3\beta-\alpha)}{\beta(\beta-\alpha)} - \left[\frac{\alpha^3(3\beta-\alpha)}{\beta^2(\beta-\alpha)} + \frac{2\alpha}{\beta}\right]\exp(\alpha\rho\sigma_\theta^2)\right\},$$

where Ω is a positive constant. We see that introducing some randomness in the monetary process will be welfare improving if this derivative is positive for $\rho = 0$, i.e. if

$$\alpha[\beta(3\alpha - 2) - \alpha^2] > 0. \tag{18}$$

This will hold for couples (α, β) such that (i) $\alpha < 0$, (ii) $\alpha > 0$ and $\beta > \alpha^2/(3\alpha - 2)$. It holds therefore in particular for $\alpha = 1$ and $\beta = 2$, the values emphasized by Polemarchakis and Weiss.

6.1. An explanation for the puzzle

In order to understand this somewhat counterintuitive result, we shall now compare the market solution to that which would be chosen by an omniscient social planner maximizing the above criterion. Maximization of $\mathrm{E}(\Theta_t U_t)$ subject to the feasibility constraints gives the following values $L_t^*$and C_t^* for period t:

$$L_t^* = \left(\frac{\Theta_{t-1}}{\Theta_t}\right)^{(1-\alpha)/(\beta-\alpha)} \qquad C_t^* = \left(\frac{\Theta_t}{\Theta_{t-1}}\right)^{(\beta-1)/(\beta-\alpha)}. \tag{19}$$

We can compare for example the value of L_t^* to the value found for the market equilibrium (formula 11 with N_{jt} replaced by Θ_t):

$$L_t = A^{1/\beta-\alpha}\left(\frac{X_t}{\Theta_t}\right)^{1-\gamma} = A^{1/\beta-\alpha}\left(\frac{X_t}{\Theta_t}\right)^{(1-\rho)\alpha/\beta}. \tag{20}$$

[4] A more detailed argument is developed in the appendix.

Comparing formulas (19) and (20), we can note a few things: First, employment in period t fails to respond to the number Θ_{t-1} of agents in the "old generation", whereas it should in the optimal solution (19). This problem cannot be fixed by monetary policy. Secondly, we see that L_t reacts somehow in an inadequate way to variations in Θ_t (compare the exponents of Θ_t in formulas 19 and 20). But there monetary policy can sometimes help. We see in particular that a more random monetary policy (i.e. an increase in ρ) will move the exponent of Θ_t in the right direction for values of α and β such that (i) $\alpha < 0$, or (ii) $\alpha > 0$ and $\alpha/\beta > (1-\alpha)/(\beta-\alpha)$, which can be summarized as $\alpha[\beta(2\alpha - 1) - \alpha^2] > 0$. Of course, this more random policy creates at the same time some unwanted "noise", so that the actual condition for an improvement (Eq. (18)) is a bit more stringent than this, but the mechanism is clear: more random money is desirable only in cases where it corrects the inadequate reaction of market equilibrium to the current Θ_t shock.

7. Lump-sum money transfers

Up to now we studied our model under the assumption that money transfers were made proportionately to money holdings. We shall now move to the more traditional (and realistic) assumption of lump-sum money transfers. As a consequence, the budget constraint of an old household born in t is

$$P_{t+1}C_{t+1} = P_t L_t + T_{t+1}, \tag{21}$$

where T_{t+1} denotes the money transfer per old household. With lump-sum transfers the average rate of monetary creation will matter, so we shall assume that $x_t = \operatorname{Log} X_t$ has a variance σ_x^2 and a mean μ_x. Because of this modification on monetary policy, it will be possible to give a closed form solution only for the subclass of utility functions (1) corresponding to $\alpha = 1$, so that the utility of generation t households is

$$U_t = C_{t+1} - \frac{L_t^\beta}{\beta}, \quad \beta > 1. \tag{22}$$

The rest of the model is identical. A household of generation t will choose L_t to maximize his expected utility subject to budget constraint (21), i.e. he will solve the following program

$$\text{Maximize } \mathrm{E}_t(C_{t+1}) - \frac{L_t^\beta}{\beta} = \mathrm{E}_t\left(\frac{P_t L_t + T_{t+1}}{P_{t+1}}\right) - \frac{L_t^\beta}{\beta},$$

whose first-order conditions are

$$L_t^{\beta-1} = \mathrm{E}_t\left(\frac{P_t}{P_{t+1}}\right). \tag{23}$$

Using the same methods as in Sections 3 and 4, we find that Eq. (23) leads to the following expressions for labor and consumption on island j:

$$L_{jt} = \Lambda^{1/\beta - 1}\left(\frac{X_t}{N_{jt}}\right)^{1-\gamma} \qquad C_{jt} = \Lambda^{1/\beta - 1}\frac{X_t^{1-\gamma}N_{jt}^{\gamma}}{N_{jt-1}}, \tag{24}$$

where

$$\rho = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_n^2 + \sigma_\theta^2} \qquad \gamma = 1 - \frac{1-\rho}{\beta}, \tag{25}$$

$$\Lambda = \exp\left\{\frac{1}{2}(\gamma^2\sigma_n^2 + \gamma^2\sigma_x^2 + \rho\sigma_n^2) - (1 - \rho + \gamma)\mu_x\right\}. \tag{26}$$

We immediately note that, except for the expression of the constant Λ which differs substantially, these expressions are exactly the same as in the "proportional money" case (compare to formulas (11)–(13) with $\alpha = 1$). This means in particular that the "positive" results of Section 5, which do not depend on the value of Λ, will also be the same. We do not repeat them here, and turn to the welfare analysis.

8. The lump-sum case: Welfare analysis

We shall again consider the original Lucas case without aggregate shocks (so that $N_{jt} = \Theta_{jt}$). Using formulas (22), (24)–(26), we can compute the expected utility of the representative household, and obtain

$$\begin{aligned} \mathrm{E}(\Theta_t U_t) = {} & \exp\left\{\frac{\sigma_\theta^2}{2(\beta-1)}\left[\frac{1}{1-\rho} + \frac{3(\beta-1)\rho}{\beta} + \frac{1}{\beta} + \beta - 3\right] - \frac{\mu_x}{\beta - 1}\right\} \\ & - \frac{1}{\beta}\exp\left\{\frac{\sigma_\theta^2}{2(\beta-1)}\left[\frac{\beta}{1-\rho} + \rho\left(2\beta - 1 - \frac{1}{\beta}\right) + \frac{1}{\beta} - 2\right] - \frac{\beta\mu_x}{\beta - 1}\right\}. \end{aligned} \tag{27}$$

We shall now look for the optimal combination of the two governmental policy parameters ρ and μ_x. We optimize in two steps: differentiating (27) with respect to μ_x, we find a first condition

$$\mu_x = \frac{\sigma_\theta^2}{2}\left[\frac{1}{1-\rho} + \frac{2(\beta-1)\rho}{\beta} - 1\right]. \tag{28}$$

Assume now that the government has already optimized in μ_x and insert the value of μ_x (28) into (27). We obtain

$$E(\Theta_t U_t) = \left(1 - \frac{1}{\beta}\right)\exp\left\{\frac{\sigma_\theta^2}{2}\left[\frac{\rho}{\beta} + \frac{\beta - 1}{\beta}\right]\right\}.$$

This is always increasing in ρ, and the optimum is thus attained for $\rho = 1$. So the optimal government policy is to take infinitely large σ_x^2 and μ_x, both values being related by Eq. (28)!

In order to explain this highly puzzling result, let us consider the first best optimum, which is still given by formula (19) for $\alpha = 1$.

$$L_t^* = 1, \qquad C_t^* = \frac{\Theta_t}{\Theta_{t-1}}. \tag{29}$$

Now we can compare this value of L_t^* to the value found for labor in market equilibrium (formula 24):

$$L_t = \Lambda^{1/\beta - 1}\left(\frac{X_t}{\Theta_t}\right)^{1-\gamma} = \Lambda^{1/\beta - 1}\left(\frac{X_t}{\Theta_t}\right)^{(1-\rho)/\beta}. \tag{30}$$

Here we see immediately the cause of this strange result: in the first best optimum, labor should somehow not respond at all to any shock. But this can be achieved in a market equilibrium only by totally blurring market signals via an infinite money variance σ_x^2 ($\rho = 1$). There is of course a negative side to this infinite variance, which is that it induces inefficiently high Λ (formula 26). But it is clear from the same formula that this can be corrected via a large enough μ_x, so that at the optimal (ρ, μ_x) combination we will have $\Lambda = 1$. In that way the market equilibrium (30) exactly mimics the first best (29).

Of course we should remember that, as compared with the "proportional money" case, we studied only the case where $\alpha = 1$, and that it is likely that with smaller values of α this result could be somewhat attenuated. But the result remains disturbing, since this parameter is by no means pathological in this class of models.

9. Conclusions

At this stage we are left with very contrasted feelings: on the one hand, the signal extraction model delivers interesting and empirically relevant "positive" results. On the other hand, the normative results are somewhat unpalatable, as they call, in nonpathological cases, to engineer highly random and inflationary monetary policy.

In view of these normative results a number of people might be tempted to forget altogether the signal extraction story. But this would be clearly throwing the baby with the bathwater. Indeed in incomplete markets economies, such as the ones we actually live in, it would be extremely farfetched to assume that all agents receive enough market signals to fully understand the economic environment relevant to them. As a consequence there are numerous situations where the signal extraction problem should be relevant, and this problem is clearly worth further study.

So a natural next step in the agenda will be to construct rigorous macroeconomic signal extraction models which, while retaining the good positive properties of the Lucas model, will deliver reasonable optimal policy prescriptions.[5]

Appendix

We develop here a more detailed argument for the probabilities used in deriving the expected utility criterion $\mathrm{E}(\Theta_t U_t)/\mathrm{E}(\Theta_t)$ in Section 6.

In each period t events occur in two steps: first a number Θ_{jt} is drawn for each island $j \in J$. Secondly the agents are allocated randomly to the islands. In this second step, the probability for an individual agent to go to island j is $\Theta_{jt}/\sum_{i \in J}\Theta_{it}$.

Call $\Pi(\Theta_{jt})$ the common probability distribution of the Θ_{jt} variables.[6] Now going back to the first step, the probability of going to an island of size Θ_t is the sum of the corresponding probabilities for all islands j, i.e.:

$$\sum_{j \in J} \mathrm{Prob}\{\Theta_{jt} = \Theta_t\} \frac{\Theta_t}{\sum_{i \in J}\Theta_{it}} = J\Pi(\Theta_t)\frac{\Theta_t}{\sum_{i \in J}\Theta_{it}}.$$

Using again the law of large numbers, we have $\sum_{i \in J}\Theta_{it}/J = \mathrm{E}(\Theta_t)$, so that the probability of going to an island of size Θ_t is finally equal to $\Theta_t\Pi(\Theta_t)/\mathrm{E}(\Theta_t)$. This is equal to the probability that Θ_t occurs, $\Pi(\Theta_t)$, multiplied by $\Theta_t/\mathrm{E}(\Theta_t)$, as indicated in Section 6.

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[5] That such a program is actually not infeasible is demonstrated in Bénassy (1997).

[6] In order to make the notation a little less clumsy, we write probabilities "as if" the variables Θ_{jt} took discrete values.

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