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The Phillips Curve and Optimal Policy in a Structural Signal Extraction Model

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We study a monetary economy subject to "signal extraction" problems, and investigate within that framework the positive and normative aspects of monetary policy. As in Lucas (1972, *Journal of Economic Theory*, **4**, 103–124; 1973, *American Economic Review*, **63**, 326–334), imperfect signal perception generates macroeconomic correlations similar to those found in the "Phillips curve" literature. Moving to normative aspects, we find that, when aggregate shocks are present, traditional nonactivist policies do not permit reaching the first best, and that an intelligent activist policy always leads to better outcomes. The specific characteristics and effectiveness of this optimal policy also depend crucially on the problem of signal extraction. *Journal of Economic Literature* Classification Number: E5. © 2001 Academic Press

1. INTRODUCTION

The purpose of this article is to construct a structural model of signal extraction which will (i) have rigorous microfoundations, (ii) display "good" positive properties (say, similar to Lucas, 1973), and (iii) lead to reasonable and intuitive policy prescriptions. As we shall see, this is less of an easy task than might have been thought initially.

Of the few stories that give some nontrivial role to money under market clearing, a most prominent one is certainly the "signal extraction" story initiated by Lucas (1972, 1973). Although these two articles had a major

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impact on the profession, they have had surprisingly little direct following. Besides the fact that many people found these models "difficult," at least two more objective reasons can be thought of.

(a) On the one hand, the 1973 model, which was the most popular in teaching, and from which most "positive" results were derived, was a model without explicit microfoundations. Although this was standard at the time, it has clearly become a problem in view of the sustained recent emphasis on microfoundations.

(b) On the other hand, the 1972 model, which indeed had rigorous microfoundations, quickly turned out to have bizarre normative implications: for some standard parameterizations of the utility functions, one finds that the optimal monetary policy consists in having random money increases, possibly accompanied by very large inflation (Polemarchakis and Weiss, 1977, Bénassy, 1999). In fact, with the same utility functions we shall be using below, optimal money increases have unbounded mean and variance! This is clearly not a policy prescription one feels comfortable with.

Confronted with these problems, a first negative reaction might be to abandon this type of model altogether. We want to argue that this would be throwing the baby out with the bath water, and we will do so in a constructive manner by building a model with the following characteristics: (i) it has rigorous foundations, of the same spirit as Lucas (1972), but with a different structure; (ii) it has the same good positive properties as Lucas (1973). For example, we shall explicitly derive a Phillips curve; (iii) finally in terms of normative properties, the policy prescriptions are much more reasonable than those of the original model. First, out of all "simple" policies where money increases have constant mean and variance, the optimal one is that where the money stock is constant from period to period. Second, we also investigate more sophisticated rules, where money reacts to past aggregate shocks, and we find that such activist rules always dominate the "constant money" one. Moreover the signal extraction problem fundamentally affects the characteristics of these optimal policies.

2. THE MODEL

2.1. *Markets and Agents*

The economy consists of a continuum of sectors, or "islands," indexed by $j \in [0,1]$. Firms in sector j are competitive and produce a specific intermediate good indexed by j with the following technology:

$$Q_{jt} = Z_{jt} L_{jt} \tag{1}$$

where Q_{jt} and L_{jt} are production and employment in sector j, and Z_{jt} the technological shock in sector j. We shall denote by P_{jt} and W_{jt} the price and wage in sector j.

Households do not consume the intermediate goods, but a unique consumption good produced by competitive firms endowed with the following Cobb-Douglas technology:

$$\log Y_t = \int_{j=0}^{1} \Theta_{jt} \log Q_{jt} dj \tag{2}$$

$$\int_j \Theta_{jt} dj = 1 \tag{3}$$

where Θ_{jt} is a "local shock" on sector j. We shall denote by P_t the price of this consumption good.

Each sector is endowed with atomistic households of mass one. The demography of these households is characterized by an overlapping generations structure (Samuelson, 1958). All households are identical across sectors, so we omit the index j in describing them. The typical household of generation t works L_t when young (in period t), and consumes C_{t+1} when old (in period $t+1$). The utility function of generation t households is:

$$U_t = C_{t+1} - \frac{L_t^\beta}{\beta} \qquad \beta > 1 \tag{4}$$

2.2. *Policy*

Since households consume one period after they have worked, they must transfer wealth from one period to the next, which is done by accumulating money, the only store of value. Call M_t the quantity of money in period t (which is entirely in the hands of old households). Government policy consists in engineering changes in this quantity of money. To be specific, aggregate money supply is multiplied between periods $t-1$ and t by a coefficient X_t:

$$M_t = X_t M_{t-1} \tag{5}$$

This money increase is carried out by making money transfers to the old households in a lump-sum manner (or taxing them if there is a decrease in the aggregate quantity of money). As a result the budget constraint for old households in sector j is:

$$P_t C_{jt} = W_{jt-1} L_{jt-1} + T_{jt} \tag{6}$$

where T_{jt} is the lump sum money transfer in period t to the old households in sector j. We assume these transfers are the same for all households, so that:

$$T_{jt} = (X_t - 1) M_{t-1} \tag{7}$$

2.3. *Stochastic Shocks and Information*

We saw above that the various sectors are subject to technology shocks Z_{jt}. These shocks have a local and a global component:

$$Z_{jt} = Z_t A_{jt} \tag{8}$$

Now let us call:

$$z_{jt} = \log Z_{jt} \qquad z_t = \log Z_t \qquad a_{jt} = \log A_{jt} \qquad \theta_{jt} = \log \Theta_{jt}$$

We shall assume that z_t and a_{jt} are normal with mean zero and variances σ_z^2 and σ_a^2. The variables θ_{jt} are also normal with variance σ_θ^2 and mean μ_θ. In view of (3), μ_θ and σ_θ^2 are related by:

$$E(\Theta_{jt}) = \exp\left(\mu_\theta + \frac{\sigma_\theta^2}{2}\right) = 1 \qquad \mu_\theta = -\frac{\sigma_\theta^2}{2} \tag{9}$$

It is further assumed that in period t households and firms in sector j know M_{t-1} and the local variables P_{jt}, Q_{jt}, W_{jt}, and L_{jt}. They do not know the aggregate variables P_t, X_t or Y_t, which they will learn about only in the following period.

3. THE OPTIMUM

We shall compute as a benchmark the trades that would be chosen by an omniscient social planner maximizing the expected utility of the representative agent. This optimum is obtained by maximizing in each period t the average across sectors of living households' utilities, equal to:

$$V_t = \int_j \left(C_{jt} - \frac{L_{jt}^\beta}{\beta}\right) dj = Y_t - \int_j \frac{L_{jt}^\beta dj}{\beta} \tag{10}$$

Maximizing V_t subject to the production function (2), we obtain:

$$L_{jt} = (\Theta_{jt} Y_t)^{1/\beta} \tag{11}$$

Insert these into the production function, and use the following equality:

$$\int_j \Theta_{jt} \log \Theta_{jt} dj = \log \zeta \qquad \zeta = \exp\left(\frac{\sigma_\theta^2}{2}\right) \tag{12}$$

We obtain the optimum values for employment and output:

$$Y_t = \zeta^{1/(\beta-1)} Z_t^{\beta/(\beta-1)} \tag{13}$$

$$L_{jt} = \zeta^{1/\beta(\beta-1)} \Theta_{jt}^{1/\beta} Z_t^{1/(\beta-1)} \tag{14}$$

We can also compute for future reference the representative household's expected utility under this optimum scheme:

$$E(V_t) = \left(1 - \frac{1}{\beta}\right) \exp\left[\left(\frac{\beta}{\beta-1}\right)^2 \frac{\sigma_z^2}{2} + \frac{1}{\beta-1}\frac{\sigma_\theta^2}{2}\right] \tag{15}$$

4. MARKET EQUILIBRIUM UNDER STOCHASTIC MONEY

We shall now assume, as in most signal extraction models, that money increases are stochastic. More precisely the logarithm of money increases $x_t = \log X_t$ has a normal distribution with variance σ_x^2 and mean μ_x. We want to find the optimal values of σ_x^2 and μ_x, and for that we shall first compute in this section the market equilibrium for a given policy.

We start with a few equilibrium relations that will hold in all equilibrium situations considered below. First, because the intermediate goods producing firms are competitive, we have:

$$W_{jt} = Z_{jt} P_{jt} \tag{16}$$

Second, since the firms producing the consumption good are competitive, the relative product price of intermediate j will be equal to its marginal productivity, so that:

$$\frac{P_{jt}}{P_t} = \frac{\Theta_{jt} Y_t}{Q_{jt}} \tag{17}$$

Third, the demand for consumption goods comes from old agents only. They entirely spend their money M_t, so the condition of equilibrium on the goods market is simply:

$$Y_t = \frac{M_t}{P_t} \tag{18}$$

Now in order to compute the equilibrium we have to find the amount of labor supplied by households. The household in sector j maximizes his expected utility:

$$\text{Max } E_{jt}\left(\frac{W_{jt} L_{jt} + T_{jt+1}}{P_{t+1}}\right) - \frac{L_{jt}^{\beta}}{\beta} \tag{19}$$

where E_{jt} denotes an expectation based on information available in sector j only. The first-order condition is:

$$L_{jt}^{\beta-1} = W_{jt} E_{jt}\left(\frac{1}{P_{t+1}}\right) \tag{20}$$

Computation of the conditional expectation in (20) turns out to be complex, and leads to the following characterization of equilibrium, derived in Appendix 2.

PROPOSITION 1. *The equilibrium under stochastic money is characterized by*:

$$L_{jt} = \Lambda^{1/(\beta-1)} \zeta^{(1-\rho)/\beta(\beta-1)} \left(\Theta_{jt} X_t\right)^{(1-\rho)/\beta} \tag{21}$$

$$Y_t = \Lambda^{1/(\beta-1)} \zeta^{(1-\rho)/(\beta-1)} Z_t X_t^{(1-\rho)/\beta} \tag{22}$$

where:

$$\rho = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\theta^2} \tag{23}$$

$$\Lambda = \exp\left[-(1-\rho+\gamma)\mu_x + \frac{\gamma^2\sigma_x^2}{2} + \frac{\sigma_z^2}{2}\right] \tag{24}$$

$$\gamma = 1 - \frac{1-\rho}{\beta} \tag{25}$$

We note that money increases have a positive effect on employment and output. The mechanism is exactly the same as in Lucas (1972). It will be best understood if we consider first the "benchmark" case where agents are fully informed about all shocks. It is easy to compute, using methods similar to those of Appendix 2, that employment in sector j is then given by:

$$L_{jt} = \exp\left[\frac{\sigma_z^2}{2(\beta-1)}\right]\zeta^{1/\beta(\beta-1)}\Theta_{jt}^{1/\beta} \tag{26}$$

Agents' labor supply reacts thus positively to local real shocks Θ_{jt}, but not at all to money shocks X_t. Now combining formulas (1), (5), (16), (17) and (18), we obtain:

$$P_{jt}Q_{jt} = W_{jt}L_{jt} = \Theta_{jt}M_t = \Theta_{jt}X_tM_{t-1} \tag{27}$$

We see that agents in sector j observe the product $\Theta_{jt}X_t$, but not the two shocks separately. We can note that the exponent $(1-\rho)/\beta$ of $\Theta_{jt}X_t$ in (21) is a combination between zero and $1/\beta$, i.e., the exponents of Θ_{jt} and X_t in (26). As we should expect, the relative weights depend crucially on the parameter ρ, which intuitively represents how much of a shock in $\Theta_{jt}X_t$ is attributable to a shock in X_t.

5. POSITIVE RESULTS AND THE PHILLIPS CURVE

Since we have obtained closed form solutions for employment and output, we can now derive a number of correlations between macroeconomic aggregates, and we shall see that, from a positive point of view, our model behaves very much like Lucas' (1973) famous model. An interesting difference is that we have supply shocks in addition to demand shocks, and this may modify substantially some of the results, as was pointed out by Wallace (1992).

Let us start by going to logarithms, which will be denoted by lowercase letters:

$$y_t = (1-\gamma)\,x_t + z_t + \omega \tag{28}$$

with:

$$\omega = \frac{1}{\beta - 1}\,[\log \Lambda + (1-\rho)\log \zeta] \tag{29}$$

Now prices are deduced from output through $p_t = m_t - y_t$, so that:

$$p_t = m_t - (1-\gamma)\,x_t - z_t - \omega \tag{30}$$

and denoting as $\pi_t = p_t - p_{t-1}$ the inflation rate:

$$\pi_t = \gamma x_t + (1-\gamma)\,x_{t-1} - (z_t - z_{t-1}) \tag{31}$$

Using Eqs. (28) and (31) we can now describe a number of positive properties of our model.

5.1. *The Phillips Curve*

Let us begin with the inflation–output correlation, whose scrutiny has generated a large amount of literature under the generic name of the "Phillips curve." To lighten formulas we shall actually compute the covariance of output and inflation, which from formulas (28) and (31) is equal to:[2]

$$\mathrm{Cov}\,(y_t, \pi_t) = \gamma\,(1-\gamma)\,\sigma_x^2 - \sigma_z^2 \tag{32}$$

In view of the relations between γ, ρ, σ_θ^2, and σ_x^2 (formulas 23 and 25), this can be rewritten as:

$$\mathrm{Cov}\,(y_t, \pi_t) = \frac{\rho\,(\beta - 1 + \rho)}{\beta^2}\,\sigma_\theta^2 - \sigma_z^2 \tag{33}$$

We first see that, if there are no supply shocks ($\sigma_z^2 = 0$), the covariance between output and inflation is positive (of course the variance of money shocks, and thus ρ, must be strictly positive). This corresponds to the traditional "Phillips curve." But we should note also that, with high enough variance of productivity shocks σ_z^2, this covariance can actually become negative (Wallace, 1992).

[2]We assume for these computations that the x_t and z_t are i.i.d.

5.2. *Output Regressions*

Let us investigate how much output reacts to money expansions by regressing output y_t on money increases x_t. From Eq. (28), the coefficient of regression is trivially:

$$1-\gamma=\frac{1-\rho}{\beta}=\frac{\sigma_\theta^2}{\beta\left(\sigma_x^2+\sigma_\theta^2\right)} \tag{34}$$

We see that, as was observed by Lucas (1973), the coefficient of regression of output on money increases is a decreasing function of the variance σ_x^2 of these money increases.

6. SHOULD THE GOVERNMENT RANDOMIZE?

We shall now start our normative investigation by considering simple policies (constant μ_x and σ_x^2), and asking whether it can be optimal here to conduct a random or inflationary monetary policy, as this is a problematic issue in the original signal extraction model (Polemarchakis and Weiss, 1977, Bénassy, 1999). The optimal policy is characterized by the following proposition.

PROPOSITION 2. *Within the category of simple policies with constant mean and variance, the optimal one is that with "constant money":*

$$\mu_x=\sigma_x^2=0 \tag{35}$$

Proof. In order to find the optimal monetary policy, we compute the expected utility of the representative household:

$$E\left(U_t\right)=E\left(\frac{W_{jt}L_{jt}+T_{jt+1}}{P_{t+1}}-\frac{L_{jt}^\beta}{\beta}\right) \tag{36}$$

Using formulas in Proposition 1, we find that this is equal to:

$$E\left(U_t\right)=\exp\left\{\frac{\beta\sigma_z^2}{2\left(\beta-1\right)}-\frac{\mu_x}{\beta-1}+\left[\left(1-\gamma\right)^2+\frac{\gamma^2}{\beta-1}\right]\frac{\sigma_x^2}{2}+\frac{\left(1-\rho\right)\sigma_\theta^2}{2(\beta-1)}\right\}$$

$$-\frac{1}{\beta}\exp\left\{\frac{\beta\sigma_z^2}{2\left(\beta-1\right)}-\frac{\beta\mu_x}{\beta-1}+\frac{\beta\gamma^2\sigma_x^2}{2\left(\beta-1\right)}+\frac{(1-\rho)\sigma_\theta^2}{2(\beta-1)}\right\} \tag{37}$$

Maximizing first with respect to μ_x, we find that the first-order condition is simply that the two quantities within brackets be equal, which yields:

$$\mu_x=\left(2\gamma-1\right)\frac{\sigma_x^2}{2} \tag{38}$$

Let us insert this value of μ_x into (37), and recall from the definition of ρ that $\sigma_x^2 = \rho\sigma_\theta^2/(1-\rho)$. We obtain the following expression:

$$E(U_t) = \left(1 - \frac{1}{\beta}\right) \exp\left\{\frac{\beta\sigma_z^2}{2(\beta-1)} + \frac{(1-\rho)(\beta+\rho)}{\beta(\beta-1)}\frac{\sigma_\theta^2}{2}\right\} \tag{39}$$

The maximum of this quantity with respect to ρ is reached for $\rho = 0$, and therefore $\sigma_x^2 = 0$. Combining this with (38), we see that the optimal policy is to have μ_x equal to zero as well, i.e., a policy of constant money. *Q.E.D.*

This normative prescription is quite reasonable, and shows that the optimality of randomizing in some versions of the original Lucas model was not an intrinsic feature of signal extraction models in general. The value of the utility obtained under this constant money policy is then computed by inserting $\rho = 0$ into (39):

$$E(U_t) = \left(1 - \frac{1}{\beta}\right) \exp\left[\frac{\beta\sigma_z^2}{2(\beta-1)} + \frac{\sigma_\theta^2}{2(\beta-1)}\right] \tag{40}$$

Comparing formula (40) with the expression of utility attained in the first best (Eq. 15), we may note two things: first, if there are no aggregate shocks, i.e., if $\sigma_z^2 = 0$, then the "constant money" policy leads to the first best. As soon as there are aggregate shocks ($\sigma_z^2 \neq 0$), however, then the above is not true anymore: the situation attained under constant money is not the first best. We shall now see whether more sophisticated policies will allow us to alleviate, even partly, this inefficiency.

7. A SOPHISTICATED ACTIVIST MONETARY POLICY

We shall now investigate more intelligent policies, where monetary policy is allowed to react to macroeconomic variables. A number of authors have actually already shown that, in incomplete markets economies such as the O.L.G. structure considered here, activist policies may dominate nonactivist ones.[3]

Why activist policies may be successful is actually intuitive if we note that, at the optimum, labor supply in each sector j should react positively to the aggregate technology shock Z_t (formula 14), whereas it does not in the nonactivist case (formula 21, for example). So we have the intuition that a policy that would make labor supply increase with Z_t could be beneficial.

[3]See, for example, Sargent and Wallace (1982), and particularly Bulow and Polemarchakis (1983).

Now we shall encounter a new and major problem since, although the agents should react to the aggregate shock Z_t, this shock is actually not observable directly. As a consequence the signal extraction problem will strongly interfere with the design of optimal policy

We shall now discuss the characteristics of our policy. Because technology shocks Z_t are the only aggregate shocks, we assume that money expansions react to Z_t only. Another very important constraint is that, following Sargent and Wallace (1975, 1976), we do not want to assume that government has better information than the private sector. So we shall assume that money in period $t+1$ will react to the shock Z_t observed in the *previous* period, which is by then public knowledge:

$$\frac{M_{t+1}}{M_t} = \Phi Z_t^{\alpha} \tag{41}$$

We note that, although money increases would be fully predictable for private agents if they knew Z_t, they will have again a signal extraction problem, since they observe only the composite shock Z_{jt}. We shall now see how this problem influences the optimal policy and the level of utility attained.

7.1. *Market Equilibrium*

As in Section 4, the household in sector j maximizes his expected utility and the first-order condition is:

$$L_{jt}^{\beta-1} = W_{jt} E_{jt}\left(\frac{1}{P_{t+1}}\right) \tag{42}$$

Again we have a complex signal extraction problem, and the equilibrium is characterized through the following proposition, proved in Appendix 3.

PROPOSITION 3. *The equilibrium under policy* (41) *is characterized by:*

$$L_{jt} = \left(\frac{\Omega}{\Phi}\right)^{1/(\beta-1)} \zeta^{1/\beta(\beta-1)} \Theta_{jt}^{1/\beta} Z_{jt}^{-\alpha\nu/\beta} \tag{43}$$

$$Y_t = \left(\frac{\Omega}{\Phi}\right)^{1/(\beta-1)} \zeta^{1/(\beta-1)} Z_t^{(\beta-\alpha\nu)/\beta} \tag{44}$$

where:

$$\nu = \frac{\sigma_z^2}{\sigma_z^2 + \sigma_a^2} \tag{45}$$

$$\Omega = \exp\left[\frac{\alpha^2(1-\nu)\sigma_z^2}{2} + \left(\frac{\beta-\alpha\nu}{\beta}\right)^2 \frac{\sigma_z^2}{2}\right] \tag{46}$$

We see (formula 43) that employment in sector j now reacts to the observed technological shock Z_{jt}. This is quite intuitive since a high Z_{jt} makes the agent perceive a high aggregate shock Z_t. If α is positive he will accordingly expect high money next period (because of policy rule 41), therefore high aggregate prices tomorrow, which will lead him to decrease labor supply (formula 42).

7.2. *Optimal Policy*

Using the results of the preceding proposition we can now characterize the best policy within the class (41).

PROPOSITION 4. *Within the class of policies $M_{t+1}/M_t = \Phi Z_t^\alpha$ the optimal one is given by*:

$$\alpha = -\frac{\beta}{\beta - \nu} \tag{47}$$

$$\Phi = \exp\left(\frac{\alpha^2 \sigma_z^2}{2}\right) \tag{48}$$

and the expected utility reached is:

$$E(U_t) = \left(1 - \frac{1}{\beta}\right) \exp\left[\frac{\beta^2 \sigma_z^2}{2(\beta - \nu)(\beta - 1)} + \frac{\sigma_\theta^2}{2(\beta - 1)}\right] \tag{49}$$

Proof. Using the values in Proposition 3, we can compute the expected utility of the representative household:

$$E(U_t) = \Phi^{-1/(\beta-1)} \zeta^{1/(\beta-1)} \exp\left[\frac{\alpha^2 (1-\nu) \sigma_z^2}{2(\beta-1)} + \frac{(\beta - \alpha\nu)^2 \sigma_z^2}{2\beta(\beta-1)}\right]$$

$$- \frac{\Phi^{-\beta/(\beta-1)}}{\beta} \zeta^{1/(\beta-1)} \exp\left[\frac{(\beta-\nu)\alpha^2 \sigma_z^2}{2(\beta-1)} + \frac{(\beta - \alpha\nu)^2 \sigma_z^2}{2\beta(\beta-1)}\right] \tag{50}$$

We want to maximize $E(U_t)$ with respect to the two arguments of the policy function, α and Φ. Starting with Φ, we first obtain $\Phi = \exp\left(\alpha^2 \sigma_z^2/2\right)$ (Eq. 48). Inserting this value of Φ into the expression of $E(U_t)$, we find:

$$E(U_t) = \left(1 - \frac{1}{\beta}\right) \zeta^{1/(\beta-1)} \exp\left[\frac{(\beta - \alpha\nu)^2 - \alpha^2 \beta\nu}{2\beta(\beta-1)} \sigma_z^2\right] \tag{51}$$

Maximizing now with respect to α, we obtain the second optimal policy parameter $\alpha = -\beta/(\beta - \nu)$ (Eq. 47), as well as the expected utility level which will be attained under this optimal policy (Eq. 49). *Q.E.D.*

We see that the parameter ν (formula 45), which is the central parameter in the signal extraction problem considered here, influences both the optimal response of money to aggregate shocks (Eq. 47) and the final utility attained (Eq. 49). We note that the level of utility attained $E(U_t)$ is always above the "constant money" level of utility (Eq. 40). Moreover the higher ν, the higher the utility gain, and for $\nu = 1$ one actually reaches the first best optimum. All this is quite intuitive since the higher ν, the more informative about Z_t are the sectoral shocks Z_{jt} observed by individual agents.

8. CONCLUSIONS

Earlier studies of signal extraction models à la Lucas (1972, 1973) had found it difficult to build a model with both rigorous microfoundations and reasonable positive and normative conclusions about monetary policy. In particular it was shown (Polemarchakis and Weiss, 1977; Bénassy, 1999) that, with the same utility functions as the ones used here, the policy prescription coming out of the model's original structure was to engineer money increases with unbounded average and variance. The purpose of this article was to construct a different model of signal extraction in order to reconcile rigorous microfoundations with reasonable positive and normative implications.

So in a first step we constructed a model of signal extraction highly similar in spirit to Lucas (1972), but with different production interrelations between islands. We also enriched the structure by having actually two signal extraction problems: one on the demand addressed to each island and one on technology shocks. In spite of this added complexity, we could obtain closed form solutions throughout the model.

We first examined the positive implications of the model, and saw notably that it could generate a Phillips curve very similar to that derived in Lucas (1973). We noted that, if supply shocks are strong enough, the positive inflation–output covariance can be turned into a negative one.

We then started our investigation of normative policies by asking whether this model can solve the problem of erratic policy prescriptions described above. It actually does. We considered indeed the set of all policies with constant mean and variance of money increases, and found that the best was the "constant money" one. This shows that signal extraction models are a perfectly viable basis for delivering sensible policy prescriptions.

Policies defined by constant mean and variance of money increases are a bit too simplistic, however, and we wanted to find out whether more sophisticated policies would allow us to do better. And the answer is clearly yes: as soon as there are aggregate supply shocks, activist policies, where money reacts adequately to past values of the supply shocks, always lead to a higher level of utility.[4] As may be expected, this utility gain will be lower, and optimal policy reaction different, when perception of aggregate shocks is blurred by idiosyncratic ones. Here also the signal extraction problem matters!

APPENDIX 1

We give here a few standard formulas which are repeatedly used below. Let us first recall the formula giving the expectation of lognormal variables. Assume that A is a normal variable with mean μ and variance σ^2. Then:

$$E(e^A) = \exp\left(\mu + \frac{\sigma^2}{2}\right) \tag{52}$$

Now assume that A and B are jointly normal with means μ_A and μ_B, variances σ_A^2 and σ_B^2, and covariance σ_{AB}. Then the stochastic variable A conditional on B is a normal variable with the following mean and variances:

$$E(A \mid B) = \mu_A + \frac{\sigma_{AB}}{\sigma_B^2}(B - \mu_B) \tag{53}$$

$$\sigma^2(A \mid B) = \sigma_A^2\left(1 - \frac{\sigma_{AB}^2}{\sigma_A^2 \sigma_B^2}\right) \tag{54}$$

APPENDIX 2

In order to evaluate the expectation $E_{jt}(1/P_{t+1})$ in (20), the household will hold a "price theory." More precisely, he will conjecture a relation of the form:

$$P_t = \xi M_{t-1} X_t^\gamma Z_t^\delta \tag{55}$$

where the parameters γ, δ and ξ will be determined below from the assumption of rational expectations. In view of (55) the expectation in (20) is rewritten:

$$E_{jt}\left(\frac{1}{P_{t+1}}\right) = \frac{1}{\xi M_{t-1}} E_{jt}\left(\frac{1}{X_t}\right) E_{jt}\left(\frac{1}{X_{t+1}^\gamma}\right) E_{jt}\left(\frac{1}{Z_{t+1}^\delta}\right) \tag{56}$$

[4] Appendix 4 shows that this result is actually not limited to the specific functional forms used in this article.

We use formula (52) above, which yields:

$$E_{jt}\left(\frac{1}{X_{t+1}^{\gamma}}\right) = \exp\left(-\gamma\mu_x + \frac{\gamma^2\sigma_x^2}{2}\right) \tag{57}$$

$$E_{jt}\left(\frac{1}{Z_{t+1}^{\delta}}\right) = \exp\left(\frac{\delta^2\sigma_z^2}{2}\right) \tag{58}$$

The point where signal extraction comes in is in the evaluation of $E_{jt}(1/X_t)$, since the household in sector j observes the product $\Theta_{jt}X_t$, but not Θ_{jt} or X_t individually. Call $\rho = \sigma_x^2/\left(\sigma_x^2 + \sigma_\theta^2\right)$. Going to logarithms and using (53) and (54), we see that, conditional on the observation of $x_t + \theta_{jt}$, x_t is a normal variable with mean and variance:

$$E\left(x_t \mid x_t + \theta_{jt}\right) = \mu_x + \rho\left(x_t + \theta_{jt} - \mu_x - \mu_\theta\right) \tag{59}$$

$$Var\left(x_t \mid x_t + \theta_{jt}\right) = \rho\sigma_\theta^2 \tag{60}$$

so that:

$$E_{jt}\left(\frac{1}{X_t}\right) = \left(\frac{1}{\Theta_{jt}X_t}\right)^{\rho} \exp\left[\rho\mu_\theta - (1-\rho)\mu_x + \frac{\rho\sigma_\theta^2}{2}\right] \tag{61}$$

and in view of Eq. (9):

$$E_{jt}\left(\frac{1}{X_t}\right) = \left(\frac{1}{\Theta_{jt}X_t}\right)^{\rho} \exp\left[-(1-\rho)\mu_x\right] \tag{62}$$

Putting (56), (57), (58) and (62) together we obtain:

$$E_{jt}\left(\frac{1}{P_{t+1}}\right) = \frac{\Lambda}{\xi M_{t-1}}\left(\Theta_{jt}X_t\right)^{-\rho} \tag{63}$$

$$\Lambda = \exp\left[-(1-\rho+\gamma)\mu_x + \frac{\gamma^2\sigma_x^2}{2} + \frac{\delta^2\sigma_z^2}{2}\right] \tag{64}$$

Now combine (63) with (27) and the first-order condition (20). This yields:

$$L_{jt} = \left[\frac{\Lambda}{\xi}\left(\Theta_{jt}X_t\right)^{1-\rho}\right]^{1/\beta} \tag{65}$$

Inserting this value into the production function, we obtain the output and price levels:

$$Y_t = \left(\frac{\Lambda}{\xi}\right)^{1/\beta} \zeta^{(1-\rho)/\beta} Z_t X_t^{(1-\rho)/\beta} \tag{66}$$

$$P_t = \frac{X_t M_{t-1}}{Y_t} = \left(\frac{\Lambda}{\xi}\right)^{-1/\beta} \frac{M_{t-1}\zeta^{-(1-\rho)/\beta} X_t^{1-(1-\rho)/\beta}}{Z_t} \tag{67}$$

Identifying (67) with the price formula (55), we find:

$$\gamma = 1 - \frac{1-\rho}{\beta} \qquad \delta = -1 \tag{68}$$

$$\xi = \Lambda^{-1/(\beta-1)} \zeta^{-(1-\rho)/(\beta-1)} \tag{69}$$

Inserting these values into (65), (66), and (64), we obtain the final values of employment and output (formulas 21, 22, and 24 in Proposition 1).

APPENDIX 3

In order to evaluate the expectation $E_{jt}(1/P_{t+1})$ in (42), the household will hold a "price theory." More precisely, he will conjecture a relation of the form:

$$P_t = \lambda M_t Z_t^{\eta} \tag{70}$$

where the parameters η and λ will be determined below from rational expectations. In view of (41) and (70) the expectation in (42) is rewritten:

$$E_{jt}\left(\frac{1}{P_{t+1}}\right) = \frac{1}{\lambda \Phi M_t} E_{jt}\left(\frac{1}{Z_t^{\alpha}}\right) E_{jt}\left(\frac{1}{Z_{t+1}^{\eta}}\right) \tag{71}$$

$$E_{jt}\left(\frac{1}{Z_{t+1}^{\eta}}\right) = \exp\left(\frac{\eta^2 \sigma_z^2}{2}\right) \tag{72}$$

The point where signal extraction comes in is in the evaluation of $E_{jt}(1/Z_t^{\alpha})$. Indeed the agents in sector j know $Z_{jt} = Z_t A_{jt}$, but not Z_t itself. Call $\nu = \sigma_z^2 / \left(\sigma_z^2 + \sigma_a^2\right)$. We go to logarithms, and see that, conditional on the observation of $z_{jt} = z_t + a_{jt}$, z_t is a normal variable with mean and variance:

$$E\left(z_t \mid z_{jt}\right) = \nu z_{jt} \tag{73}$$

$$Var\left(z_t \mid z_{jt}\right) = (1-\nu)\,\sigma_z^2 \tag{74}$$

so that:

$$E_{jt}\left(\frac{1}{Z_t^{\alpha}}\right) = \left(\frac{1}{Z_{jt}}\right)^{\alpha\nu} \exp\left[\frac{\alpha^2(1-\nu)\sigma_z^2}{2}\right] \tag{75}$$

and finally:

$$E_{jt}\left(\frac{1}{P_{t+1}}\right)=\frac{\Omega}{\lambda\Phi M_t}\left(\frac{1}{Z_{jt}}\right)^{\alpha\nu} \tag{76}$$

$$\Omega=\exp\left[\frac{\alpha^2(1-\nu)\sigma_z^2+\eta^2\sigma_z^2}{2}\right] \tag{77}$$

Combine (76) with (27) and the first-order condition (42):

$$L_{jt}=\left[\frac{\Omega\Theta_{jt}}{\lambda\Phi}\left(\frac{1}{Z_{jt}}\right)^{\alpha\nu}\right]^{1/\beta} \tag{78}$$

Now inserting this value into the production function, we obtain:

$$Y_t=\left(\frac{\Omega\zeta}{\lambda\Phi}\right)^{1/\beta}Z_t^{(\beta-\alpha\nu)/\beta} \tag{79}$$

$$P_t=\frac{M_t}{Y_t}=\left(\frac{\Omega\zeta}{\lambda\Phi}\right)^{-1/\beta}\frac{M_t}{Z_t^{(\beta-\alpha\nu)/\beta}} \tag{80}$$

By identification of (80) with the price formula (70) we find:

$$\eta=\frac{\alpha\nu}{\beta}-1 \qquad \lambda=\left(\frac{\Phi}{\Omega\zeta}\right)^{1/(\beta-1)} \tag{81}$$

Inserting these values into (78), (79), and (77), we obtain the final values of employment and output (formulas 43, 44, and 46 in Proposition 3).

APPENDIX 4

We want to show now that the results on the optimality of policy activism we obtained are not a "knife edge" phenomenon, but remain valid when the model is subject to various perturbations. As an example, we shall show in this appendix that a useful simplifying assumption of our model, the linearity of utility in consumption, can be dispensed with. To that effect we shall generalize our setting to utility functions of the form:

$$U_t=F(C_{t+1},\varepsilon)-\frac{L_t^\beta}{\beta} \tag{82}$$

where $F(C_{t+1},\varepsilon)$ is a family of utility functions indexed on a parameter ε, concave in C_{t+1}, continuous in ε, and such that:

$$F(C_{t+1},0)=C_{t+1} \tag{83}$$

This family of functions "nests" the one we have been using in the main text. To get an intuitive idea we may use as an example the family $F(C_{t+1}, \varepsilon) = C_{t+1}^{1-\varepsilon}$. Now let us investigate again policies of the form $M_{t+1}/M_t = \Phi Z_t^{\alpha}$. Of course we shall not be able to give an exact value for the optimal α, but the solution must be of the form:

$$\alpha = \phi(\varepsilon) \tag{84}$$

and we know that for $\varepsilon = 0$ the optimal parameter α takes the value that we have computed in Section 7, i.e.:

$$\phi(0) = -\frac{\beta}{\beta - \nu} \tag{85}$$

Since the problem is continuous, we know that, for values of ε not too far from 0, $\phi(\varepsilon)$ will be close to the above value, and thus different from 0, so that the optimal policy will also be an activist one.

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