

# Continuous Implementation\*

Marion Oury

Olivier Tercieux

University of Cergy-Pontoise

Paris School of Economics and CNRS

## Abstract

In this paper, we introduce a notion of continuous implementation and characterize when a social choice function is continuously implementable. More specifically, we say that a social choice function is continuously (partially) implementable if it is (partially) implementable for types in the model under study and it continues to be (partially) implementable for types "close" to this initial model. Our results show that this notion is tightly connected to full implementation in rationalizable strategies.

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## 1 Introduction

The notion of partial – as opposed to full – implementation consists in designing games under which *some* equilibrium – but not necessarily *all* – yields the outcome desired by the social planner. Despite the fact that undesirable equilibria may potentially exist, partial implementation is widely used both in theoretical and applied works. The main contribution of this paper is to give a precise sense in which even if the social planner is only

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willing to partially implement the social choice function, a lack of full implementation may make the predictions based on partial implementation very unrobust to slight perturbations of the information structure. Taking into account the doubts a social planner may have about his model, we characterize when a social choice function can be partially *continuously* implemented.<sup>1</sup>

We consider a situation where the social planner (i) is willing to *partially* implement in strict Nash equilibrium and (ii) wants to implement in a continuous manner. More specifically, we require that in any model that embeds the initial model, there exists an equilibrium that (i) is a strict equilibrium on the initial model, (ii) yields the desired outcome, not only at all types of the initial model but also at all types “close” to initial types. Of course, the notion of closeness in types is critical. To formalize this, we use the method introduced by Harsanyi (1967) and developed in Mertens and Zamir (1985). Each type in the initial model is mapped into a hierarchy of beliefs. Then, following the interim approach due to Weinstein and Yildiz (2007), we define a notion of “nearby” types. This notion, formally described by the product topology in the universal type space, captures the restrictions on the modeler’s ability to observe the players’ (high order) beliefs.

Our main results state that this continuity requirement leads to necessary conditions that are tightly linked to full implementation in rationalizable strategies. More specifically, we first show that a social choice function can be partially continuously implemented only if it is monotonic. Monotonicity was first introduced by Maskin (1999) in a complete information setting and was then generalized to the incomplete information setting by Postlewaite and Schmeidler (1986) and further studied in Jackson (1991). While partial implementation is often a quite weak requirement, monotonicity is known to have a lot of bite.<sup>2</sup> It is necessary for full implementation in Nash equilibrium and it is sufficient under weak additional conditions when there are more than three players. Hence, our result builds a first bridge between partial and full implementation. We then generalize this

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<sup>1</sup>Bergemann and Morris (2005) is one of the first attempts to relax the implicit common knowledge assumptions made in the mechanism design literature. In their setting, the modeler when choosing a mechanism has no information on the real situation that will finally prevail among agents. Consequently, their notion of robust implementation follows a “global approach”: a social choice function is robustly (partially) implementable if it is (partially) implementable *on all possible models*. In contrast, we assume that the planner has some specific model in mind and is quite confident about it. As a consequence, our requirement is local: the social choice function must be implemented only at types “close” to types in the initial model.

<sup>2</sup>For instance, under complete information, with more than three players, any social choice function can be partially implemented while on the universal domain of preferences only constant social choice functions are monotonic (see Saijo (1987)).

result and show that interim rationalizable monotonicity<sup>3</sup> – a generalization of Bayesian monotonicity – is also a necessary condition for partial continuous implementation. Since, with more than three players, some weak domain restrictions are enough for any interim rationalizable monotonic social choice function to be fully implementable in rationalizable strategies, we get a connection between partial continuous implementation and full implementation in rationalizable strategies.

Finally, we discuss how far our conditions are from sufficiency. More precisely, we drop the assumption that the planner is willing to partially implement in strict equilibrium (but keep the standard requirement of pureness) and slightly relax the assumption that messages are cheap-talk (sending a message may involve an arbitrarily small cost). We provide a full characterization of partial continuous implementation: a social choice function is continuously implementable by a finite mechanism if and only if it is fully implementable in rationalizable messages.

Since the seminal paper by Rubinstein (1989) on the e-mail game, several approaches<sup>4</sup> have been followed to analyze the connection between high order beliefs and strategic behavior. These works share the common assumption that in the perturbed models, some types may have preferences over action profiles that are radically different from those of types in the initial model<sup>5</sup>. Indeed, the behavior of these specific types is used as a starting point for contagion processes that drive results in these analyses. Note that the meaning of such an assumption in the mechanism design context (where the social planner fixes the game form) is problematic. However, in the present paper, we show that the logic of implementation makes this assumption unnecessary.<sup>6</sup> Indeed, in mechanism design, several different states of nature are ex ante possible for the social planner. Our argument in the proof uses this multiplicity and shows that this setting is then rich enough: partial implementation in the initial model is used as an (endogenous) starting point for the contagion process at equilibrium.

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<sup>3</sup>This notion is an interim formulation of a notion introduced by Bergemann and Morris (2011) in the ex post setting. It was developed in discussions between D. Bergemann, S. Morris and O. Tercieux and is the relevant necessary condition for interim rationalizable implementation. This can be shown using the types of arguments in Bergemann and Morris (2011) and in Bergemann, Morris and Tercieux (2011).

<sup>4</sup>The notion of *robustness* due to Kajii and Morris (1997), the *global games* argument due to Carlsson and Van Damme (1993) and the *interim approach* due to Weinstein and Yildiz (2007).

<sup>5</sup>This corresponds to the notion of "crazy types" in the robustness approach and to that of "dominance regions" in the global games or in the interim approach.

<sup>6</sup>In particular, our two main theorems will be derived with no assumption on the space of states of nature which is in stark contrast with the richness assumption of Weinstein and Yildiz (2007) or its weakening proposed by Chen (2010).

## 2 Setting

We consider a finite set of players  $\mathcal{I} = \{1, \dots, I\}$ . Each agent  $i$  has a bounded utility function  $u_i : A \times \Theta \rightarrow \mathbb{R}$  where  $\Theta$  is the countable set of states of nature and  $A$  is the set of outcomes. A mechanism specifies a message set  $M_i$  for each agent  $i$  and a mapping  $g$  from message profiles to outcomes. More precisely, we write  $M$  as an abbreviation for  $\prod_{i \in \mathcal{I}} M_i$  and for each player  $i$ ,  $M_{-i}$  for  $\prod_{j \neq i} M_j$ .<sup>7</sup> A mechanism  $\mathcal{M}$  is a pair  $(M, g)$  where  $M$  is countable and the outcome function  $g : M \rightarrow A$  assigns to each message profile  $m$  an alternative  $g(m) \in A$ . In general, if  $X$  is a topological space, we treat it as a measurable space with its Borel sigma-field and the space of Borel probability measures on  $X$  is denoted  $\Delta(X)$ . Spaces  $\Delta(X)$  are endowed with the topology of weak convergence of measures. Unless otherwise specified, if  $X$  is countable, it is treated as a topological space endowed with the discrete topology. The only countable spaces that we may sometimes treat differently are type spaces; this point is detailed further in the text. The space of outcomes  $A$  is not necessarily countable and is assumed to be a Hausdorff topological space. By a slight abuse of notations, given a space  $X$ , we will sometimes note  $x$  for the degenerate distribution in  $\Delta(X)$  assigning probability 1 to  $\{x\}$ ;  $g$  will also be extended to measures i.e., given  $\alpha \in \prod_{i \in \mathcal{I}} \Delta(M_i)$ , we abuse notations and note  $g(\alpha)$  for the measure  $\alpha \circ g^{-1}$ . Similarly, utility functions  $u_i$ 's will be extended to measures over  $A$ .

### Models

There are two main classes of situations with incomplete information. The first one consists in situations with an ex ante stage during which each player observes a private signal about the payoffs, and the joint distribution of signals and payoffs is commonly known. These situations are naturally modelled using a standard type space. The second class, on which we focus in this paper, consists in genuine situations of incomplete information, i.e., situations with no ex ante stage: each player begins with a hierarchy of beliefs. We follow the standard Harsanyi (1967)'s approach and model these hierarchies of beliefs by introducing a hypothetical ex ante stage leading to a standard type space. This allows us to study strategic behavior of players at types that are considered to be close to a given original model.

A model  $\mathcal{T}$  is a pair  $(T, \kappa)$  where  $T = T_1 \times \dots \times T_I$  is a countable<sup>8</sup> type space which is a complete and separable topological space and  $\kappa(t_i) \in \Delta(\Theta \times T_{-i})$  denotes the associated beliefs for each  $t_i \in T_i$ . We also note  $\kappa(t_i)[E]$  for the probability of any measurable set

<sup>7</sup>Similar abbreviations will be used throughout the paper for analogous objects.

<sup>8</sup>This assumption is made to ensure existence of Bayes Nash equilibrium in finite games which will turn out to be useful when we deal with sufficient conditions for continuous implementation. See also footnote 34 in the supplementary material section.

$E \subset \Theta \times T_{-i}$  given by  $\kappa(t_i)$ . The function  $\kappa(t_i)$  is assumed to be continuous with respect to  $t_i$  for each player  $i$ .<sup>9</sup> In what follows, for two models  $\mathcal{T} = (T, \kappa)$  and  $\mathcal{T}' = (T', \kappa')$ , we will note  $\mathcal{T} \supset \mathcal{T}'$  if  $T \supset T'$ , the sigma field on each  $T'_i$  is the relative sigma field obtained from each  $T_i$  and for all  $i, t_i \in T'_i : \kappa(t_i)[E] = \kappa'(t_i)[(\Theta \times T'_{-i}) \cap E]$  for any measurable  $E \subset \Theta \times T_{-i}$ . Given a mechanism  $\mathcal{M}$  and a model  $\mathcal{T}$ , we write  $U(\mathcal{M}, \mathcal{T})$  for the induced incomplete information game. In this game, a (behavioral) strategy of a player  $i$  is any measurable function  $\sigma_i : T_i \rightarrow \Delta(M_i)$ . We will note  $\sigma_i(m_i | t_i)$  for the probability that strategy  $\sigma_i$  assigns to message  $m_i$  when player  $i$  is of type  $t_i$ . For each  $i \in \mathcal{I}$  and for each belief  $\pi_i \in \Delta(\Theta \times M_{-i})$ , we set

$$BR_i(\pi_i | \mathcal{M}) = \arg \max_{m_i \in M_i} \sum_{(\theta, m_{-i}) \in \Theta \times M_{-i}} \pi_i(\theta, m_{-i}) [u_i(g(m_i, m_{-i}), \theta)].$$

Since we allow for infinite mechanism, this set may be empty. Given any type  $t_i$  and any strategy profile  $\sigma_{-i}$ , we write  $\pi_i(\cdot | t_i, \sigma_{-i}) \in \Delta(\Theta \times M_{-i})$  for the joint distribution on the underlying uncertainty and the other players' messages induced by  $t_i$  and  $\sigma_{-i}$ .

**Definition 1** *A profile of strategies  $\sigma = (\sigma_1, \dots, \sigma_I)$  is a Bayes Nash equilibrium in  $U(\mathcal{M}, \mathcal{T})$  if for each  $i \in \mathcal{I}$  and each  $t_i \in T_i$ ,*

$$m_i \in \text{Supp}(\sigma_i(t_i)) \Rightarrow m_i \in BR_i(\pi_i(\cdot | t_i, \sigma_{-i}) | \mathcal{M}).$$

Given two models  $\mathcal{T}$  and  $\mathcal{T}'$  such that  $\mathcal{T} \supset \mathcal{T}'$ , and given a strategy  $\sigma$  in  $U(\mathcal{M}, \mathcal{T})$ , we note  $\sigma|_{\mathcal{T}'}$  for the strategy  $\sigma$  restricted to  $\mathcal{T}'$ . It is easily checked that if  $\sigma$  is a Bayes Nash equilibrium in  $U(\mathcal{M}, \mathcal{T})$ , then  $\sigma|_{\mathcal{T}'}$  is also a Bayes Nash equilibrium in  $U(\mathcal{M}, \mathcal{T}')$ .

Given a model  $(T, \kappa)$  and any type  $t_i$  in type space  $T_i$ , we can compute the belief of  $t_i$  on  $\Theta$  by setting

$$h_i^1(t_i) = \text{marg}_{\Theta} \kappa(t_i),$$

which is called the *first-order belief* of  $t_i$ . Similarly, we can compute the second-order belief of  $t_i$ , i.e., his beliefs about  $(\theta, h_1^1(t_1), \dots, h_I^1(t_I))$  by setting

$$h_i^2(t_i)[F] = \kappa(t_i)[\{(\theta, t_{-i}) | (\theta, h_1^1(t_1), \dots, h_I^1(t_I)) \in F\}]$$

for each measurable  $F \subset \Theta \times \Delta(\Theta)^I$ . Proceeding in this way, we can compute an entire hierarchy of beliefs. Hence, a type of a player  $i$  induces an infinite hierarchy of beliefs  $(h_i^1(t_i), h_i^2(t_i), \dots, h_i^\ell(t_i), \dots)$  where  $h_i^1(t_i) \in \Delta(\Theta)$  is a probability distribution

<sup>9</sup>Given that  $T$  is countable, the discrete topology makes  $T$  complete and separable. It also makes  $\kappa(t_i)$  a continuous function w.r.t.  $t_i$  for each player  $i$ . In general, we allow any topology on  $T$  that ensures these properties.

on  $\Theta$ , representing the beliefs of  $i$  about  $\theta$ ,  $h_i^2(t_i) \in \Delta(\Theta \times \Delta(\Theta)^I)$  is a probability distribution representing the beliefs of  $i$  about  $\theta$  and the others' first order beliefs, and so on... Let us write  $h_i(t_i)$  for the resulting hierarchy and  $h_i^\ell(t_i)$  for the  $\ell$ th-order beliefs of type  $t_i$ .

The set of all belief hierarchies for which it is common knowledge that the beliefs are coherent (i.e., each player knows his beliefs and his beliefs at different orders are consistent with each other) is the universal type space (see Mertens and Zamir (1985) and Brandenburger and Dekel (1993)). We denote by  $T_i^*$  the set of player  $i$ 's hierarchies of beliefs in this space and write  $T^* = \prod_{i \in I} T_i^*$ .

In our formulation, two types  $t$  and  $\bar{t}$  are close if there exists a sufficiently large  $\ell$  such that for each  $\ell' \leq \ell$ , the  $\ell'$ th-order beliefs  $h^{\ell'}(t)$  and  $h^{\ell'}(\bar{t})$  are close in the topology of convergence of measures. To be more precise, each  $T_i^*$  is endowed with the product topology, so that a sequence of types  $\{t_i[n]\}_{n=0}^\infty$  converges to a type  $t_i$ , if, for each  $\ell : h_i^\ell(t_i[n]) \rightarrow h_i^\ell(t_i)$  (i.e.,  $h_i^\ell(t_i[n])$  converges toward  $h_i^\ell(t_i)$  in the topology of weak convergence of measures<sup>10</sup>). In such a case, we write  $t_i[n] \rightarrow_P t_i$ .

We are now in a position to introduce the notion of continuous implementation. The social planner considers a model and wants to analyze how strategic behavior is affected under his mechanism when the assumption that his model is common knowledge is relaxed. In the sequel, we fix a finite model  $\bar{T} = (\bar{T}, \bar{\kappa})$  which is the model the planner has in mind. A social choice function<sup>11</sup> (SCF, henceforth) is a mapping  $f : \bar{T}_0 \rightarrow A$  where  $\bar{T}_0 \subset \bar{T}$ . In the sequel,  $\bar{T}_0$  will be interpreted as the set of profiles of types that the planner cares about. As will become clear, in some natural cases (e.g. complete information settings), this may be a proper subset of  $\bar{T}$ .<sup>12</sup>

Again, in this paper, we focus on partial implementation. We will be considering two notions of partial implementation, depending on the solution concept used, that yield two notions of continuous implementation. Traditionally, implementation theory has focused on pure Nash equilibria and a SCF  $f$  is said to be partially implementable by a mechanism  $\mathcal{M}$  if in the Bayesian game  $U(\mathcal{M}, \bar{T})$ , there is a pure Bayes Nash equilibrium that induces  $f$ . We will also consider a stronger notion of partial implementation – namely, strict partial implementation – replacing pure Bayes Nash equilibrium in the previous sentence

<sup>10</sup>Recall that  $h_i^k(t_i[n]) \in \Delta(X_{k-1})$  where  $X_0 = \Theta$  and  $X_k = [\Delta(X_{k-1})]^I \times X_{k-1}$ .

<sup>11</sup>In the paper, we restrict our attention to social choice functions for simplicity. For social choice sets, there are several notions of partial implementation one may use. In any cases, our results apply in an obvious way, see footnote 20.

<sup>12</sup>For instance, Jackson (1991) assumes that each agent has a (common support) prior over  $\bar{T}$ .  $\bar{T}_0$  is then defined as the set of profiles of types to which agents assign strictly positive probability.

by strict Bayes Nash equilibrium.<sup>13</sup> Informally, a SCF will be said to be (resp. strictly) continuously implementable if for any model containing the original benchmark model  $\bar{\mathcal{T}}$ , there is a Bayes Nash equilibrium that (i) is pure (resp. strict) on the original model i.e., we maintain the original requirement of partial implementation in pure (resp. strict) equilibrium. As we will explain, this ensures that (resp. strict) continuous implementation is a refinement of (resp. strict) partial implementation. In addition, the Bayes Nash equilibrium of the model containing the original benchmark model  $\bar{\mathcal{T}}$  must be so that (ii) if a type  $t$  is close to some type  $\bar{t}$  of the original model, then the outcome provided at  $t$  is close to the desired outcome  $f(\bar{t})$ , i.e., we impose a continuity requirement. Such an equilibrium will be said to continuously implement  $f$ . Formally,

**Definition 2** Fix a mechanism  $\mathcal{M}$  and a model  $\mathcal{T}$  such that  $\bar{\mathcal{T}} \subset \mathcal{T}$ . We say that an equilibrium  $\sigma$  in  $U(\mathcal{M}, \mathcal{T})$  (resp. strictly) continuously implements  $f : \bar{T}_0 \rightarrow A$  if (i)  $\sigma|_{\bar{\mathcal{T}}}$  is a pure (resp. strict) Nash equilibrium in  $U(\mathcal{M}, \bar{\mathcal{T}})$  (ii) for any  $\bar{t} \in \bar{T}_0$  and any sequence  $t[n] \rightarrow_P \bar{t}$  where for each  $n : t[n] \in \mathcal{T}$ , we have  $(g \circ \sigma)(t[n]) \rightarrow f(\bar{t})$ .<sup>14</sup>

We now state a formal definition of continuous implementation.

**Definition 3** A social choice function  $f : \bar{T}_0 \rightarrow A$  is (resp. strictly) continuously implementable if there exists a mechanism  $\mathcal{M}$  such that for each model  $\mathcal{T}$  with  $\bar{\mathcal{T}} \subset \mathcal{T}$ , there is an equilibrium  $\sigma$  in  $U(\mathcal{M}, \mathcal{T})$  which (resp. strictly) continuously implements  $f$ .<sup>15</sup>

As already stated, the notion of (resp. strict) continuous implementation is a refinement of (resp. strict) partial implementation. Indeed, if a social choice function is (resp. strictly) continuously implementable, then (taking  $\mathcal{T} = \bar{\mathcal{T}}$  and using (i) above), there is a pure (resp. strict) equilibrium  $\sigma$  in  $U(\mathcal{M}, \bar{\mathcal{T}})$  satisfying  $(g \circ \sigma)(\bar{t}) = f(\bar{t})$  for any  $\bar{t} \in \bar{T}_0$  i.e., the mechanism  $\mathcal{M}$  must (resp. strictly) partially implements  $f$ .

<sup>13</sup>Strict Nash equilibria resist most standard refinements in game theory. The planner may hence naturally be willing to strengthen his requirement from partial implementation in pure Nash to partial implementation in strict Nash equilibrium. We note that in some standard contexts like complete information settings, partial implementation in strict Nash equilibrium can be achieved for any social choice functions when there are two or more players and when transfers are allowed.

<sup>14</sup>The function  $g \circ \sigma$  is the composition of the mapping  $\sigma : T \rightarrow \prod_{i \in \mathcal{I}} \Delta(M_i)$  and of the mapping  $g : \prod_{i \in \mathcal{I}} \Delta(M_i) \rightarrow \Delta(A)$ .

<sup>15</sup>Our continuous implementation notion can be defined with other topologies than the product topology. Two natural candidates are the uniform topology or the strategic topology (see Dekel, Fudenberg and Morris (2006)). With these topologies, as expected, very permissive results can be shown. In particular, partial implementation in strict equilibrium with a finite mechanism is a sufficient condition for strict continuous implementation.

*On the complete information benchmark.* Our formalism allows  $\bar{T}_0$  to be a proper subset of  $\bar{T}$ . In other terms, the planner may not care about achieving the right outcome at all types in  $\bar{T}$ . One natural setting in which this will be the case is the complete information setting. In this situation, given a finite set of states of nature  $\Theta$ , whenever the true state is  $\theta$ , it is assumed to be common belief among agents. To incorporate this in our setting, we define the complete information model  $\mathcal{T}^{CI} = (T^{CI}, \kappa)$  as follows: for each player  $i$ ,  $T_i^{CI} = \bigcup_{\theta \in \Theta} \{t_{i,\theta}\}$  and  $\kappa(t_{i,\theta}) = \delta_{(\theta, t_{-i,\theta})}$  for all  $\theta$ , with  $\delta_{(\theta, t_{-i,\theta})}$  the probability distribution that puts probability 1 on  $\{(\theta, t_{-i,\theta})\}$ . Here, at the profile of types  $t_\theta = (t_{i,\theta})_{i \in \mathcal{I}}$ , players believe with probability one that state  $\theta$  is the true state. They all believe that the others believe that state  $\theta$  is the true state and so on... Hence, state  $\theta$  is common belief among agents. Henceforth, such profiles of types will be called complete information profiles of types. We consider that the model  $\bar{T}$  the planner has in mind is  $\mathcal{T}^{CI}$ . Note that there are profiles of types in  $\bar{T}$  that are not of complete information. Recall that, by definition, the planner is willing to achieve the right outcome only at complete information profiles of types and at profiles of types that are close to these profiles of types. In our formalism, this means that  $\bar{T}_0$  is equal to  $\bigcup_{\theta \in \Theta} \{t_\theta\}$ , which is a proper subset of  $\bar{T}$ .

*On SCFs defined on payoff-relevant types.* Most of the mechanism design literature assumes that the social choice function depends only on states of nature (and not directly on profiles of types of agents). Our approach in that respect is more general. Consider a social choice function  $f$  which now maps  $\Theta$  into  $A$ . A simple way to incorporate this formulation into our setting is to assume the existence of a mapping  $\bar{\theta} : \bar{T} \rightarrow \Theta$  which for each profile of types specifies the state of nature that the planner associates to it. Then define  $\tilde{f} : \bar{T} \rightarrow A$  by  $\tilde{f}(t) = f(\bar{\theta}(t))$  for all  $t \in \bar{T}$ . All our results can be applied to this standard setting.

In the next section, we assume that the planner considers partial implementation in strict Nash equilibrium in his benchmark model and so requires the SCF to be strictly continuously implemented. In Section 4, we will relax the assumption that a continuous equilibrium must be strict on the original model and show that similar results can be obtained provided that we slightly depart from the assumption that messages are costless.

### 3 Main Results

The first subsection below presents our basic argument in the complete information benchmark. This allows a presentation of the core of our argument with very few technicalities.

The second subsection treats the incomplete information case and provides our main results.

### 3.1 A simple illustration of the argument under complete information

In the sequel,  $\bar{T}_0$  is assumed to be equal to  $\bigcup_{\theta \in \Theta} \{t_\theta\}$ . In this section, we show that any social choice function  $f : \bar{T}_0 \rightarrow A$  that is strictly continuously implementable satisfies a strengthening of the well-known monotonicity condition defined in Maskin (1999). To explain in words this condition, it states that in case the desired alternative differs at profiles of types  $t_\theta$  and  $t_{\theta'}$ , there must exist at least one agent who, if the true state were  $\theta'$  and he expected other agents to claim the state is  $\theta$ , could be offered a reward  $a$  that would give him a strict incentive to “report” the deviation of other agents, where the reward  $a$  would not tempt him if the true state was in fact  $\theta$ , i.e., he would have a (strict) incentive to “report truthfully”. So (strict) Maskin monotonicity essentially ensures the elimination of some undesirable equilibria. Formally,

**Definition 4** *The social choice function  $f : \bar{T}_0 \rightarrow A$  is strict Maskin monotonic<sup>16</sup> if for every pair of states  $\theta$  and  $\theta'$*

1.  $f(t_\theta) = f(t_{\theta'})$  whenever

$$u_i(f(t_\theta), \theta) > u_i(a, \theta) \Rightarrow u_i(f(t_\theta), \theta') \geq u_i(a, \theta'), \text{ for all } i \text{ and } a; \quad (\star)$$

or, equivalently,

2.  $f(t_\theta) \neq f(t_{\theta'})$  implies

$$u_i(f(t_\theta), \theta) > u_i(a, \theta) \text{ and } u_i(a, \theta') > u_i(f(t_\theta), \theta'), \text{ for some } i \text{ and } a.$$

The following result, which is a first step toward our main result, reduces the gap between strict continuous implementation and full implementation in Nash equilibrium since – as established by Maskin – the above monotonicity condition is “almost” sufficient for full implementation. The strict version of Maskin monotonicity was introduced in Bergemann, Morris and Tercieux (2011) where it was shown to be tightly connected to full implementation in rationalizable strategies.<sup>17</sup>

<sup>16</sup>Maskin’s original definition is weaker: the strict inequality in  $(\star)$  is replaced by a weak one.

<sup>17</sup>Maskin (1999) showed that with more than three players together with the assumption that  $f$  satisfies the weak condition of No Veto Power, monotonicity actually implies full implementation in Nash equilibrium. In addition, with more than three players, under an economic condition and restricting attention to social choice functions that are responsive (i.e. in distinct states it selects distinct outcomes), Bergemann, Morris and Tercieux (2011) showed that full implementation in rationalizable strategies of a strict monotonic social choice function can be achieved.

**Theorem 1** *The social choice function  $f$  is strictly continuously implementable only if it is strict Maskin monotonic.*

**Proof of Theorem 1.** Assume that there exists a mechanism  $\mathcal{M} = (M, g)$  that strictly continuously implements  $f$ . Pick  $t_\theta, t_{\theta'} \in \bar{T}_0$  such that for each player  $i$  and for each  $a \in A$ , the relation  $(\star)$  is satisfied. We want to show that  $f(t_\theta) = f(t_{\theta'})$ .

We show that there exists a model  $\mathcal{T} = (T, \kappa) \supset \mathcal{T}^{CI}$  such that for any equilibrium  $\sigma$  that strictly continuously implements  $f$ , there is a sequence of types  $\{t[n]\}_{n=1}^\infty$  in  $T$  such that  $t[n] \rightarrow_P t_{\theta'}$  and  $(g \circ \sigma)(t[n]) \rightarrow f(t_\theta)$ . By point (ii) of Definition 2:  $(g \circ \sigma)(t[n]) \rightarrow f(t_{\theta'})$ , which implies  $f(t_\theta) = f(t_{\theta'})$ .

For this purpose, we build the desired model  $\mathcal{T} = (T, \kappa)$  in which for each player  $i$ , each set  $T_i$  satisfies:

$$T_i = T_i^{CI} \cup \left( \bigcup_{n=1}^{\infty} t_i[n] \right)$$

where  $t_i[n]$  and  $\kappa$  are defined recursively as follows. First,  $t_i[1]$  is such that

$$\begin{aligned} \text{marg}_{T_{-i}} \kappa(t_i[1])[t_{-i, \theta}] &= 1, \\ \text{marg}_{\Theta} \kappa(t_i[1])[\theta'] &= \frac{1}{2}, \end{aligned}$$

and

$$\text{marg}_{\Theta} \kappa(t_i[1])[\theta] = \frac{1}{2}.$$

In addition, for each  $n \geq 2$ ,  $t_i[n]$  is defined by

$$\begin{aligned} \text{marg}_{T_{-i}} \kappa(t_i[n])[t_{-i}[n-1]] &= 1, \\ \text{marg}_{\Theta} \kappa(t_i[n])[\theta'] &= 1 - \frac{1}{n+1} \end{aligned}$$

and

$$\text{marg}_{\Theta} \kappa(t_i[n])[\theta] = \frac{1}{n+1}.$$

Observe that each  $T_i$  is countable, and so is  $T$ .

Now pick any equilibrium  $\sigma$  of the induced game  $U(\mathcal{M}, \mathcal{T})$  that strictly continuously implements  $f$ . By point (i) in Definition 2,  $\sigma|_{T^{CI}}$  is a strict Nash equilibrium in  $U(\mathcal{M}, \bar{T})$  and hence,  $\sigma(t_\theta)$  is a strict Nash equilibrium in the game of complete information where payoffs are given for each player  $i$  by  $u_i(\cdot, \theta)$ . In addition, point (ii) in Definition 2 implies that  $(g \circ \sigma)(t_\theta) = f(t_\theta)$ . In the sequel, we note  $m_i^* := \sigma_i(t_{i, \theta})$  and  $m^* := \sigma(t_\theta)$ . We have for any player  $i$  and  $m'_i \neq m_i$ :

$$u_i(g(m^*), \theta) > u_i(g(m'_i, m_{-i}^*), \theta),$$

and so,

$$u_i(f(t_\theta), \theta) > u_i(g(m'_i, m^*_{-i}), \theta).$$

By  $(\star)$ , this implies that

$$u_i(f(t_\theta), \theta') \geq u_i(g(m'_i, m^*_{-i}), \theta'),$$

which in turn yields

$$u_i(g(m^*), \theta') \geq u_i(g(m'_i, m^*_{-i}), \theta'),$$

i.e.,  $m^*$  is a pure Nash equilibrium in the game of complete information where payoffs are given for each player  $i$  by  $u_i(\cdot, \theta')$ . Now, by construction, under the equilibrium  $\sigma$ , type  $t_i[1]$  believes with probability one that the other players play  $m^*_{-i}$ . He assigns strictly positive weight only to state  $\theta$ , under which  $m^*_i$  is the unique best response against  $m^*_{-i}$ , and to state  $\theta'$ , under which  $m^*_i$  is a best response against  $m^*_{-i}$ . Hence, we must have:

$$\sigma_i(t_i[1]) = m^*_i.$$

Using a similar reasoning, it is easy to show inductively that for all  $n \geq 2$

$$\sigma_i(t_i[n]) = m^*_i.$$

This means that for each  $n \geq 1$ ,  $(g \circ \sigma)(t[n]) = g(m^*) = f(t_\theta)$  and so obviously,  $(g \circ \sigma)(t[n]) \rightarrow f(t_\theta)$  (as  $n \rightarrow \infty$ ) which completes the proof since  $t[n] \rightarrow_P t_{\theta'}$  (as  $n \rightarrow \infty$ ). ■

In the next section, we generalize the above result to the Bayesian setting. The argument relies on a general necessary condition (Lemma 1 below) for strict continuous implementation which is implicitly used in the above proof. Let us briefly explain how the argument works. In the above proof, we established that strict continuous implementation implies the existence of a strict Nash equilibrium  $\sigma_{|\bar{T}}$  in  $U(\mathcal{M}, \bar{T})$  such that if  $m^*$  is a Nash equilibrium at state  $\theta'$  (i.e., in the game of complete information where payoffs are given for each player  $i$  by  $u_i(\cdot, \theta')$ ) and  $m^*$  is played by  $\sigma_{|\bar{T}}$  at  $t_\theta$ , we must have  $g(m^*) = f(t_{\theta'})$ . Note that by a similar argument, one can show that strict continuous implementation implies the existence of a strict Nash equilibrium  $\sigma_{|\bar{T}}$  in  $U(\mathcal{M}, \bar{T})$  such that if  $m^*$  is a Nash equilibrium at state  $\theta'$  and  $m^* \in \sigma_{|\bar{T}}(\bar{T})$ , we must have  $g(m^*) = f(t_{\theta'})$ .<sup>18</sup> Since this property is true for any  $\theta' \in \Theta$ , strict continuous implementation implies the existence of a strict Nash equilibrium  $\sigma_{|\bar{T}}$  in  $U(\mathcal{M}, \bar{T})$  such that if a pure equilibrium  $\tilde{\sigma}$  in  $U(\mathcal{M}, \bar{T})$  satisfies  $\tilde{\sigma}(\bar{T}) \subset \sigma_{|\bar{T}}(\bar{T})$ , we must have  $g(\tilde{\sigma}(\bar{t})) = f(\bar{t})$  for all  $\bar{t} \in \bar{T}_0$ . As we will see, this condition is also necessary for strict continuous implementation in the Bayesian setting (Lemma 1) and implies the usual monotonicity condition of the social choice function.

<sup>18</sup>Here, given an equilibrium  $\sigma$ ,  $\sigma(\bar{T})$  stands for the image of  $\sigma$ , i.e.  $\bigcup_{\bar{t} \in \bar{T}} \sigma(\bar{t})$ .

### 3.2 The general argument under incomplete information

We now drop the restriction that the benchmark model  $\bar{T}$  is a complete information model. In the sequel, to make the argument simpler, we assume that the planner cares about all profiles of types in  $\bar{T}$ , i.e.,  $\bar{T}_0 = \bar{T}$ . All definitions provided below can easily be adapted to the case where  $\bar{T}_0$  is a proper subset of  $\bar{T}$  (as, for instance, in Jackson (1991)). All our results as well can be extended in a straightforward way.

Our results show a tight connection between the notion of partial continuous implementation and full implementation in (interim correlated) rationalizable strategies. Let us first recall the definition of (interim correlated) rationalizability given in Dekel, Fudenberg and Morris (2006, 2007). Fix a mechanism  $\mathcal{M}$  and a type space  $\mathcal{T}$ . Let  $\Sigma_i : T_i \rightarrow 2^{M_i} \setminus \{\emptyset\}$  be a specification of possible messages for each type of player  $i$  and let  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$ .

**Definition 5** *Given a mechanism  $\mathcal{M}$  and a type space  $\mathcal{T}$ ,  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  is a best-reply set in  $U(\mathcal{M}, \mathcal{T})$  if for each  $t_i$  and  $m_i \in \Sigma_i(t_i)$ , there exists  $\pi_i \in \Delta(\Theta \times T_{-i} \times M_{-i})$  where  $\text{marg}_{\Theta \times T_{-i}} \pi_i = \kappa(t_i)$ ,  $\pi_i(\theta, t_{-i}, m_{-i}) > 0 \implies m_{-i} \in \Sigma_{-i}(t_{-i})$  and  $m_i \in BR_i(\text{marg}_{\Theta \times M_{-i}} \pi_i \mid \mathcal{M})$ .*

For each player  $i$ , the set of rationalizable messages at type  $\bar{t}_i$ , denoted  $R_i(\bar{t}_i \mid \mathcal{M}, \bar{T})$ , is the set of messages  $m_i$  such that  $m_i \in \Sigma_i(\bar{t}_i)$  for some best-reply set  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$ . We also note  $R(\bar{t} \mid \mathcal{M}, \bar{T})$  for  $\prod_{i \in \mathcal{I}} R_i(\bar{t}_i \mid \mathcal{M}, \bar{T})$ . We say that a SCF  $f$  is implementable in rationalizable strategies (or, rationalizable implementable) on  $\bar{T}$  if there is a mechanism  $\mathcal{M}$  under which for all  $\bar{t} \in \bar{T}$ , (1)  $R(\bar{t} \mid \mathcal{M}, \bar{T}) \neq \emptyset$  and (2)  $m \in R(\bar{t} \mid \mathcal{M}, \bar{T}) \implies g(m) = f(\bar{t})$ . In the sequel, an equilibrium  $\sigma$  will be considered as a single-valued best-reply set. Also, given a best-reply set  $\Sigma$ , we will note  $\Sigma(\bar{T})$  for the image of  $\Sigma$  i.e.,  $\bigcup_{\bar{t} \in \bar{T}} \Sigma(\bar{t})$ .

As already mentioned, our argument in the proof of Theorem 1 shows that strict continuous implementation implies the existence of a strict equilibrium  $\bar{\sigma}$  on  $\bar{T}$ , under which any equilibrium  $\sigma$  satisfying  $\sigma(\bar{T}) \subset \bar{\sigma}(\bar{T})$ , must also satisfy  $g(\sigma(\bar{t})) = f(\bar{t})$  for all  $\bar{t} \in \bar{T}$ . The result below shows that not only this must be true for any equilibrium  $\sigma$  satisfying  $\sigma(\bar{T}) \subset \bar{\sigma}(\bar{T})$ , but it is also true for best-reply sets. This lemma will generalize to the incomplete information setting the idea of the contagion argument used in the proof of Theorem 1. Our main results will be derived from this lemma.

**Lemma 1** *Assume  $f : \bar{T} \rightarrow A$  is strictly continuously implementable by a mechanism  $\mathcal{M}$ . There exists a strict equilibrium  $\bar{\sigma}$  in  $U(\mathcal{M}, \bar{T})$  such that if a best-reply set  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  in  $U(\mathcal{M}, \bar{T})$  satisfies  $\Sigma(\bar{T}) \subset \bar{\sigma}(\bar{T})$ , then  $g(\Sigma(\bar{t})) = \{f(\bar{t})\}$  for all  $\bar{t} \in \bar{T}$ .*

**Proof.** See Appendix. ■

Let us relate Lemma 1 to Weinstein and Yildiz (2004, 2007). To make the comparison clearer, let us fix a (finite) game and say that a pure equilibrium  $\bar{\sigma}$  in the model  $\bar{T}$  is continuous if for each model  $\mathcal{T} \supset \bar{T}$ , there is an equilibrium  $\sigma$  such that for any sequence  $t[n] \rightarrow_P \bar{t} \in \bar{T}$  (where for each  $n : t[n] \in \mathcal{T}$ ), we have  $\sigma(t[n]) \rightarrow \bar{\sigma}(\bar{t})$ .

Mimicking the argument in Weinstein and Yildiz (2004, 2007), it is easy to show that, in *generic games*<sup>19</sup>, if a pure equilibrium  $\bar{\sigma}$  in the model  $\bar{T}$  has full range (i.e., for each player  $i$  and each action of this player, there is a type in  $\bar{T}_i$  who plays this action under  $\bar{\sigma}$ ), and if in addition, this equilibrium is continuous, then we must have  $R(\bar{t} | \bar{T}) = \{\bar{\sigma}(\bar{t})\}$  for all  $\bar{t} \in \bar{T}$ . Otherwise stated, if  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  is a best-reply set in the model  $\bar{T}$ , then  $\Sigma(\bar{t}) = \{\bar{\sigma}(\bar{t})\}$  for all  $\bar{t} \in \bar{T}$ . One important point behind this result is that if the equilibrium  $\bar{\sigma}$  has full range, and if there is a best-reply set  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  in the model  $\bar{T}$  with  $\Sigma(\bar{t}) \neq \{\bar{\sigma}(\bar{t})\}$  for some  $\bar{t} \in \bar{T}$ , then, because the action profiles played in the best-reply set  $\Sigma$  (i.e., the action profiles in  $\Sigma(\bar{T})$ ) are also played under  $\bar{\sigma}$  at some profile of types (i.e.,  $\Sigma(\bar{T}) \subset \bar{\sigma}(\bar{T})$ ), we can use a contagion argument to exhibit a model  $\mathcal{T}$  and a sequence  $t[n]$  in this model that converges toward  $\bar{t}$  for which any equilibrium  $\sigma$  (that coincides with  $\bar{\sigma}$  on  $\bar{T}$ ) is such that  $\sigma(t[n])$  does not converge toward  $\bar{\sigma}(\bar{t})$ , i.e., showing that  $\bar{\sigma}$  is not continuous.

Given this observation, one simple generalization of Weinstein and Yildiz (2004, 2007) would state that in *generic games*, if a pure equilibrium  $\bar{\sigma}$  (that need not have full range) in the model  $\bar{T}$  is continuous, then we must have that, if  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  is a best-reply set in the model  $\bar{T}$  and  $\Sigma(\bar{T}) \subset \bar{\sigma}(\bar{T})$ , then  $\Sigma(\bar{t}) = \{\bar{\sigma}(\bar{t})\}$  for all  $\bar{t} \in \bar{T}$ .

An important difference between our setup and that of Weinstein and Yildiz (2004, 2007) is that, in our mechanism design setting, the game is not part of the primitives: it is an endogenous object fixed by the social planner. Hence, we cannot make the assumption that the game is generic. Nevertheless, the above lemma shows that if  $f$  is strictly continuously implementable, then a necessary condition similar to that exhibited above for the existence of a continuous equilibrium in the model  $\bar{T}$  arises.

We now show that the necessary condition for strict continuous implementation exhibited in Lemma 1 implies Bayesian monotonicity. This notion was first introduced by Postlewaite and Schmeidler (1986) followed by Jackson (1991). We first recall its definition.

**Definition 6** *A deception is a collection of mappings  $\beta = (\beta_i)_{i=1}^I$  where each  $\beta_i : \bar{T}_i \rightarrow 2^{\bar{T}_i}$ . A deception  $\beta$  is acceptable if for all  $t \in \bar{T}$ ,  $t' \in \beta(t) \Rightarrow f(t') = f(t)$ . Deception  $\beta$  is unacceptable if it is not acceptable.*

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<sup>19</sup>More precisely, when the set of strict rationalizable action profiles is equal to the set of rationalizable action profiles in  $\bar{T}$ , which is true for a generic choice of payoffs.

**Definition 7** *The social choice function  $f : \bar{T} \rightarrow A$  is Bayesian monotonic on  $(\bar{T}, \bar{\kappa})$  if for every unacceptable single-valued deception  $\beta : \bar{T} \rightarrow \bar{T}$ , there exist  $i, t_i \in \bar{T}_i$  and  $y : \bar{T} \rightarrow A$  such that*

$$\sum_{\theta, \tilde{t}_{-i}} \bar{\kappa}(t_i)[\theta, \tilde{t}_{-i}] u_i((y \circ \beta)(t_i, \tilde{t}_{-i}), \theta) > \sum_{\theta, \tilde{t}_{-i}} \bar{\kappa}(t_i)[\theta, \tilde{t}_{-i}] u_i((f \circ \beta)(t_i, \tilde{t}_{-i}), \theta)$$

while

$$\sum_{\theta, \tilde{t}_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, \tilde{t}_{-i}] u_i(f(\tilde{t}_i, \tilde{t}_{-i}), \theta) \geq \sum_{\theta, \tilde{t}_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, \tilde{t}_{-i}] u_i(y(\beta_i(\tilde{t}_i), \tilde{t}_{-i}), \theta)$$

for all  $\tilde{t}_i \in \bar{T}_i$ . The social choice function  $f : \bar{T} \rightarrow A$  is strict Bayesian monotonic on  $(\bar{T}, \bar{\kappa})$  if the above inequality is strict whenever  $\tilde{t}_i = \beta_i(t_i)$ .

To explain in words Bayesian monotonicity and how it relates to full implementation in Bayes Nash equilibrium, let us briefly interpret the different conditions of the definition. As for Maskin monotonicity, Bayesian monotonicity ensures elimination of some undesirable equilibria. For instance, consider an equilibrium  $\sigma$  which achieves the social choice function. If agents use an unacceptable (single-valued) deception  $\beta$ , then the strategy profile  $\sigma \circ \beta$  should not be an equilibrium (since it does not achieve the social choice function). Now, the existence of  $y$  (in the definition of Bayesian monotonicity) with the stated properties allows agent  $i$  to signal that  $\beta$  is played: the agent is rewarded according to  $y$  which makes him strictly better-off at type  $t_i$ . The other part of the Bayesian monotonicity condition ensures that agent  $i$  cannot gain by falsely accusing the other agents of using deception  $\beta$ .

Our first main result is stated as follows.

**Theorem 2** *The social choice function  $f$  is strictly continuously implementable on  $(\bar{T}, \bar{\kappa})$  only if it is strict Bayesian monotonic on  $(\bar{T}, \bar{\kappa})$ .<sup>20</sup>*

**Proof.** Let  $\mathcal{M}$  be a mechanism that strictly continuously implements  $f$ . We first make a simple remark: since a strict equilibrium  $\bar{\sigma}$  in  $U(\mathcal{M}, \bar{T})$  can be considered as a single-valued best-reply set, Lemma 1 implies that for the strict equilibrium  $\bar{\sigma}$  in  $U(\mathcal{M}, \bar{T})$

<sup>20</sup>For a social choice set, one may say that it is partially implementable if it has an element that is partially implementable. Our results then extend in an obvious way: a social choice set is strictly continuously partially implementable if it has some element that satisfies Bayesian monotonicity. We note that this is a bit different from the requirement that the social choice set itself is Bayesian monotonic in the sense of Jackson (1991). One may also say that a social choice set is partially implementable if there is a mechanism  $\mathcal{M}$  under which, for any element of the set, there is an equilibrium that coincides with this selection. In that case, a necessary condition for strict continuous partial implementation of a social choice set would be Jackson's Bayesian monotonicity.

defined in this lemma, we must have  $(g \circ \bar{\sigma})(\bar{t}) = f(\bar{t})$  for all  $\bar{t} \in \bar{T}$ . Now, pick this strict equilibrium  $\bar{\sigma}$ . Consider a single-valued deception  $\beta : \bar{T} \rightarrow \bar{T}$  such that  $(f \circ \beta)(\bar{t}) \neq f(\bar{t})$  for some  $\bar{t} \in \bar{T}$  (i.e.,  $\beta$  is unacceptable). Let  $\hat{\sigma}$  be defined by  $\hat{\sigma} = \bar{\sigma} \circ \beta$ . By definition of a deception,  $\beta(\bar{T}) \subset \bar{T}$  and hence,  $\hat{\sigma}(\bar{T}) \subset \bar{\sigma}(\bar{T})$ . Consequently, since  $\hat{\sigma}$  can be considered as a single-valued correspondence and  $(g \circ \hat{\sigma})(\bar{t}) = (g \circ \bar{\sigma} \circ \beta)(\bar{t}) = (f \circ \beta)(\bar{t}) \neq f(\bar{t})$ , by Lemma 1,  $\hat{\sigma}$  cannot be an equilibrium on  $(\bar{T}, \bar{\kappa})$ . Given this, to show the existence of  $i$ ,  $t_i \in \bar{T}_i$  and  $y : \bar{T} \rightarrow A$  satisfying the conditions in the definition of Bayesian monotonicity, we just have to mimic the usual argument: there exist  $i$ ,  $t_i \in \bar{T}_i$  and some message  $\tilde{m}_i \neq \hat{\sigma}_i(t_i)$  such that

$$\sum_{\theta, \tilde{t}_{-i}} \bar{\kappa}(t_i)[\theta, \tilde{t}_{-i}] u_i(g(\tilde{m}_i, \hat{\sigma}_{-i}(\tilde{t}_{-i})), \theta) > \sum_{\theta, \tilde{t}_{-i}} \bar{\kappa}(t_i)[\theta, \tilde{t}_{-i}] u_i(g(\hat{\sigma}(t_i, \tilde{t}_{-i})), \theta).$$

Now, let  $y : \bar{T} \rightarrow A$  be such that  $y(\bar{t}) = g(\tilde{m}_i, \bar{\sigma}_{-i}(\tilde{t}_{-i}))$  for all  $\bar{t} \in \bar{T}$ . We obtain

$$\sum_{\theta, \tilde{t}_{-i}} \bar{\kappa}(t_i)[\theta, \tilde{t}_{-i}] u_i((y \circ \beta)(t_i, \tilde{t}_{-i}), \theta) > \sum_{\theta, \tilde{t}_{-i}} \bar{\kappa}(t_i)[\theta, \tilde{t}_{-i}] u_i((f \circ \beta)(t_i, \tilde{t}_{-i}), \theta).$$

Since  $\bar{\sigma}$  is an equilibrium in  $U(\mathcal{M}, \bar{T})$  and  $(g \circ \bar{\sigma})(\bar{t}) = f(\bar{t})$  for all  $\bar{t} \in \bar{T}$ , it follows that for all  $\tilde{t}_i \in \bar{T}_i$

$$\sum_{\theta, \tilde{t}_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, \tilde{t}_{-i}] u_i(g(\bar{\sigma}(\tilde{t}_i, \tilde{t}_{-i})), \theta) \geq \sum_{\theta, \tilde{t}_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, \tilde{t}_{-i}] u_i(g(\tilde{m}_i, \bar{\sigma}_{-i}(\tilde{t}_{-i})), \theta) \quad (1)$$

which yields

$$\sum_{\theta, \tilde{t}_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, \tilde{t}_{-i}] u_i(f(\tilde{t}_i, \tilde{t}_{-i}), \theta) \geq \sum_{\theta, \tilde{t}_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, \tilde{t}_{-i}] u_i(y(\beta_i(\tilde{t}_i), \tilde{t}_{-i}), \theta) \quad (2)$$

for all  $\tilde{t}_i \in \bar{T}_i$ . Note that since  $\tilde{m}_i \neq \hat{\sigma}_i(t_i) = \bar{\sigma}_i(\beta_i(t_i))$  and since  $\bar{\sigma}$  is a strict equilibrium, for  $\tilde{t}_i = \beta_i(t_i)$ , the inequality (1) is strict and hence, inequality (2) is also strict. This proves that  $f$  is strict Bayesian monotonic on  $(\bar{T}, \bar{\kappa})$ . ■

The previous result shows that in so-called economic environments defined in Jackson (1991) and under the usual additional conditions that ensure that Bayesian monotonicity is sufficient for implementation, strict continuous implementation yields full implementation in Bayes Nash equilibrium.<sup>21</sup> Lemma 1 actually allows us to go one step further and to prove that under some weak economic conditions, strict continuous implementation

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<sup>21</sup>Recall that a SCF  $f$  is fully implementable in Bayesian Nash equilibrium if there is a mechanism that partially implements  $f$  and for every Bayes Nash equilibrium  $\sigma : \bar{T} \rightarrow M$  of the induced game  $g \circ \sigma = f$ . Bayesian monotonicity, incentive compatibility and no-veto-power of a SCF  $f$  are enough to ensure full implementation of  $f$  in economic environments. This need not be true in general environments, see Jackson (1991).

yields full implementation in rationalizable strategies. To show this, we now move to the notion of interim rationalizable monotonicity, a Bayesian version of a notion introduced in Bergemann and Morris (2011) in the ex-post environment.

**Definition 8** *The social choice function  $f : \bar{T} \rightarrow A$  is interim rationalizable monotonic on  $(\bar{T}, \bar{\kappa})$  if for every unacceptable deception  $\beta$ , there exist  $i, t_i, t'_i \in \beta_i(t_i)$  such that for every  $\psi_i \in \Delta(\Theta \times \bar{T}_{-i} \times \bar{T}_{-i})$  which satisfies the following two properties:*

- [1]  $\psi_i(\theta, t_{-i}, t'_{-i}) > 0 \Rightarrow t'_j \in \beta_j(t_j)$  for each  $j \neq i$ ,
- [2]  $\sum_{t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) = \bar{\kappa}(t_i)[\theta, t_{-i}]$  for each  $(\theta, t_{-i})$ ,

there exists  $y^* : \bar{T}_{-i} \rightarrow A$  such that

$$\sum_{\theta, t_{-i}, t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) u_i(y^*(t'_{-i}), \theta) > \sum_{\theta, t_{-i}, t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) u_i(f(t'_i, t'_{-i}), \theta)$$

and,

$$\sum_{\theta, t_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, t_{-i}] u_i(f(\tilde{t}_i, t_{-i}), \theta) \geq \sum_{\theta, t_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, t_{-i}] u_i(y^*(t_{-i}), \theta)$$

for each  $\tilde{t}_i \in \bar{T}_i$ . The social choice function  $f : \bar{T} \rightarrow A$  is strict interim rationalizable monotonic on  $(\bar{T}, \bar{\kappa})$  if the above inequality is strict whenever  $\tilde{t}_i = t'_i$ .

Let us here again briefly explain in words interim rationalizable monotonicity and how it relates to rationalizable implementation. Its role for rationalizable strategy profiles is very close to the role played for equilibria by Bayesian monotonicity. Interim rationalizable monotonicity ensures elimination of some undesirable best-reply sets. For instance, if  $\Sigma$  is a best-reply set containing rationalizable strategy profiles that all achieve the social choice function, then if agents use an unacceptable deception  $\beta$ , then  $\Sigma \circ \beta$  should not be a best-reply set (since it does not achieve the social choice function). Now, interim rationalizable monotonicity ensures that for some type  $t_i$ , for any belief of this type which is consistent with the other agents using the unacceptable deception  $\beta$ , there exists a function  $y^*$  with the stated properties which allows  $t_i$  to signal that  $\beta$  is played: the agent is rewarded according to  $y^*$  and this makes him strictly better-off. As for Bayesian monotonicity, the other part of the interim rationalizable monotonicity condition ensures that agent  $i$  cannot gain by falsely accusing the other agents of using deception  $\beta$ .

It is easy to check that if a SCF is interim rationalizable monotonic then it is Bayesian monotonic. We now state our third theorem which shows that our argument relating the necessary condition in Lemma 1 to the notion of Bayesian monotonicity actually extends to the more stringent notion of interim rationalizable monotonicity.

**Theorem 3** *If  $f : \bar{T} \rightarrow A$  is strictly continuously implementable on  $(\bar{T}, \bar{\kappa})$  then  $f$  is strict interim rationalizable monotonic on  $(\bar{T}, \bar{\kappa})$ .*

**Proof.** See Appendix. ■

As for Bayesian monotonicity, under a weak domain restriction, a SCF that is interim rationalizable monotonic is rationalizable implementable (via a stochastic mechanism i.e., a mechanism which associates lotteries over outcomes to each profile of messages) with more than three players. Hence, Theorem 3 makes a connection between strict continuously implementation and rationalizable implementation. Let us formally state this result.

Fix  $\bar{T} = (\bar{T}, \bar{\kappa})$  and recall that  $\Delta(A)$  is the set of lotteries over the set of outcomes  $A$  (which is assumed to be countable). Let us fix a social choice function  $f$ . We define the economic condition that will be needed. In order to do so, let  $Y_i$  be the set of mappings from types of player  $i$ 's opponents to lotteries satisfying the following property. Whatever agent  $i$ 's actual type is, he would never prefer the outcome to be selected according to a mapping in  $Y_i$  to what he would obtain under the social choice function  $f$  if other agents were reporting truthfully. Formally,

$$Y_i := \left\{ y : \bar{T}_{-i} \rightarrow \Delta(A) \left| \sum_{\theta, t_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, t_{-i}] u_i(f(\tilde{t}_i, t_{-i}), \theta) \geq \sum_{\theta, t_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, t_{-i}] u_i(y(t_{-i}), \theta) \text{ for all } \tilde{t}_i \in \bar{T}_i \right. \right\}.$$

**Assumption 1** *For all  $i$ , there exists  $\bar{y}_i : \bar{T}_{-i} \rightarrow \Delta(A)$  such that the following holds. For any  $\psi_i \in \Delta(\Theta \times \bar{T}_{-i})$ , there exists  $y(\cdot) \in Y_i$  satisfying*

$$\sum_{\theta, t'_{-i}} \psi_i(\theta, t'_{-i}) u_i(y(t'_{-i}), \theta) > \sum_{\theta, t'_{-i}} \psi_i(\theta, t'_{-i}) u_i(\bar{y}_i(t'_{-i}), \theta).$$

The above assumption says that if we consider preferences of player  $i$  over mappings from  $\bar{T}_{-i}$  to  $\Delta(A)$ , then there is a mapping  $\bar{y}_i : \bar{T}_{-i} \rightarrow \Delta(A)$  that, whatever player  $i$ 's beliefs over  $\Theta \times \bar{T}_{-i}$  are, some mapping in  $Y_i$  is preferred to  $\bar{y}_i$ . This assumption is satisfied in several natural environments like the one where there exist transfers or bad outcomes that are not desired by the social planner. More generally, it is also naturally satisfied by the appropriate Bayesian version of the no-worst-alternative condition as defined in Cabrales and Serrano (2009) or Bergemann, Morris and Tercieux (2011).<sup>22</sup> The sufficiency result is stated as follows.

**Proposition 1** *Assume  $I \geq 3$ . If  $f : \bar{T} \rightarrow A$  satisfies interim rationalizable monotonicity and if Assumption 1 holds, then  $f$  is interim rationalizable implementable with a stochastic mechanism.*

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<sup>22</sup>No-worst-alternative condition requires that an agent never gets his worst outcome under the social choice function.

**Proof.** See Appendix. ■

Proposition 1 together with Theorem 3 yields the following corollary.

**Corollary 1** *Assume  $I \geq 3$ . If  $f : \bar{T} \rightarrow A$  is strictly continuously implementable and if Assumption 1 holds, then  $f$  is interim rationalizable implementable.*

## 4 Continuous implementation with small costs of messages

So far, we presented necessary conditions for strict continuous implementation that are close to rationalizable implementation. Two questions are of obvious interest: first, can we relax the requirement of strict continuous implementation to continuous implementation? And then, how far are the necessary conditions from sufficiency? Let us make two simple remarks. On the first hand, it is easily shown that continuous implementation by itself is not enough to get our previous theorems.<sup>23</sup> On the other hand, given the result by Dekel, Fudenberg and Morris (2006) that the correspondence of (interim correlated) rationalizable strategies is upper hemi-continuous in the product topology in the universal type space, we know that if a SCF  $f$  is rationalizable implementable by a finite mechanism and the set of rationalizable strategies is a singleton, then  $f$  is actually strictly continuously implementable. In the sequel, to address the two questions mentioned above, we depart from the setting introduced so far by slightly relaxing the assumption that messages are costless. Restricting our attention to finite mechanisms, we obtain a full characterization result for continuous implementation (as opposed to strict continuous implementation). More precisely, we show that a SCF is continuously implementable by a finite mechanism if and only if it is implementable in rationalizable strategies by a finite mechanism. This will allow us to derive corollaries on virtual implementation which uses finite mechanisms.

### Small costs of messages

Sending a message usually requires to fill in a questionnaire, to write a letter or to make public some private information (like the willingness to pay for an object). All these actions are not costless for the agents and a recent literature in implementation takes into account potential costs of messages<sup>24</sup>. In this section, the social planner does not exclude

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<sup>23</sup>To see this, consider the simple example where for any player  $i$ , state  $\theta$  and outcome  $a$ ,  $u_i(a, \theta) = 0$ . Fix any model  $\bar{T}$  and any non-constant social choice function  $f$ . Consider a (direct) mechanism where each player is asked to report his type (i.e.  $M_i = \bar{T}_i$  for each player  $i$ ) and the outcome provided for a profile of messages  $m$  is  $f(m)$ . Since players are totally indifferent, for any model, any profile of strategy is a Bayes Nash equilibrium. Hence, it is easily checked that  $f$  is continuously implemented by the above mechanism. However, since players are totally indifferent and  $f$  is non-constant,  $f$  is not Bayesian monotonic on  $\bar{T}$ .

<sup>24</sup>See, for instance, Deneckere (2008), Matsushima (2008) and Kartik and Tercieux (2009).

that some (arbitrarily) small costs of messages may exist even if in the benchmark model messages are costless. In order to formalize this idea, we proceed as follows.

Given a mechanism  $\mathcal{M} = (M, g)$ , for each player  $i$ , we define a cost function  $c_i : M_i \times \tilde{\Theta} \rightarrow \mathbb{R}_+$  where  $\tilde{\Theta}$  is the space of states of nature associated with costs of messages. We assume that  $\tilde{\Theta}$  is rich enough. More precisely, it is defined by

$$\tilde{\Theta} = \bigcup_{i \in \mathcal{I}} \bigcup_{m_i \in M_i} \{\tilde{\theta}^{m_i}\} \cup \{\tilde{\theta}^0\}$$

where for each player  $i$  and each message  $m_i$ , we have  $c_i(m_i, \tilde{\theta}^0) = 0$ ,  $c_i(m_i, \tilde{\theta}^{m_i}) = 0$  and  $c_i(m'_i, \tilde{\theta}^{m_i}) = \eta$  for all  $m'_i \neq m_i$ , where  $\eta$  is a strictly positive parameter that can be chosen arbitrarily close to 0.<sup>25</sup> Note that since  $M$  is finite,  $\tilde{\Theta}$  is also finite. Next, we write  $\Theta^* = \Theta \times \tilde{\Theta}$  for the extended set of states of nature. For a given state of nature  $\theta^* = (\theta, \tilde{\theta}) \in \Theta^*$ , the utility function of player  $i$  for a given message  $m_i$  and a given outcome  $a$  is  $u_i(a, \theta) - c_i(m_i, \tilde{\theta})$ . Recall that, in the benchmark model of the social planner, there is no cost of messages. In other terms, the model  $\bar{\mathcal{T}} = (\bar{T}, \bar{\kappa})$  is assumed to be such that for any  $\bar{t}_i \in \bar{T}_i$ ,  $\text{marg}_{\bar{\Theta}\bar{\kappa}(\bar{t}_i)}[\tilde{\theta}^0] = 1$ . All definitions given in previous sections, are naturally extended to this setup where the space of states of nature is  $\Theta^*$ . In particular, the definition of continuous implementation (see Definitions 2 and 3) can be maintained: a social choice function  $f : \bar{T} \rightarrow A$  is continuously implementable<sup>26</sup> if there exists a mechanism  $\mathcal{M}$  such that for each model  $\mathcal{T}$  with  $\bar{\mathcal{T}} \subset \mathcal{T}$ , there is an equilibrium  $\sigma$  in  $U(\mathcal{M}, \mathcal{T})$  which continuously implements  $f$ .

Let us briefly discuss our assumption on cost of messages. Technically, this construction is used to break ties: if a type is indifferent between several messages, we can slightly perturb his information so that this type has a unique best reply. This assumption is reminiscent of the richness assumption used in Weinstein and Yildiz (2007). Indeed, if one fixes a mechanism, the richness assumption would state that for any player  $i$  and any message  $m_i$ , there exists a state of nature where  $m_i$  is strictly dominant for player  $i$ , which is much stronger than our assumption. However, as we already mentioned, in Weinstein and Yildiz (2007), the game is a primitive of the model which is not the case in our context where the mechanism is chosen by the social planner. Consequently, our construction, while logically fine, is harder to assess. Nevertheless, in the literature on partial implementation – which is the main focus of our paper – only direct mechanisms (i.e., mechanisms where the message space  $M_i$  is equal to  $\bar{T}_i$  for each player  $i$ ) are usually considered. Indeed, because of the revelation principle, this is without loss of generality

<sup>25</sup>Abusing notations, we omit the dependence of the space  $\tilde{\Theta}$  (and of the costs function  $c_i(\cdot, \cdot)$ ) with respect to the given mechanism  $\mathcal{M}$ . We also omit the dependence with respect to the parameter  $\eta$ .

<sup>26</sup>For simplicity, we maintain the assumption that  $\bar{T}_0 = \bar{T}$ .

for the characterization of incentive compatible SCFs. Direct mechanisms can naturally be considered as part of the primitives.

### Full Characterization Result

In the present setting, we obtain a full characterization result for finite mechanisms stated in Theorem 4 below. Its proof is available in the supplementary material. The proof of the necessity part remains valid for arbitrary (countable) mechanisms, but the sufficiency part does not extend to infinite countable.<sup>27</sup>

**Theorem 4** *A social choice function  $f$  is continuously implementable by a finite mechanism if and only if it is rationalizable implementable by a finite mechanism.*<sup>28</sup>

Theorem 4 allows to derive corollaries when we relax the requirement of exact implementation to virtual implementation. Given that the virtual implementation literature uses finite mechanisms, it is natural to see how Theorem 4 can be applied there. In the sequel, we assume that  $A$  is a metric space and note  $d$  the associated metric. Given a social choice function  $f$ , for each  $\delta > 0$ , we note  $B_\delta(f) = \{f' : \bar{T} \rightarrow A : d(f'(\bar{t}), f(\bar{t})) < \delta \text{ for all } \bar{t} \in \bar{T}\}$ . Following Abreu and Matsushima (1992a), we say that a social choice function  $f$  is partially virtually implementable by finite mechanisms if for each  $\delta > 0$ , there exists a social choice function  $f' \in B_\delta(f)$  that is partially implementable by a finite mechanism (that may depend on  $\delta$ ). In the same way, we can extend the definition of continuous implementation.

**Definition 9** *A social choice function  $f$  is virtually continuously implementable by finite mechanisms if for all  $\delta > 0$ , there exists a social choice function  $f' \in B_\delta(f)$  that is continuously implementable by a finite mechanism (that may depend on  $\delta$ ).*

We also say that a social choice function  $f$  is virtually rationalizable implementable by finite mechanisms if for all  $\delta > 0$ , there exists a social choice function  $f' \in B_\delta(f)$  that is rationalizable implementable by a finite mechanism. Proposition 2 below directly follows from Theorem 4.

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<sup>27</sup>We built an example of a countable infinite mechanism having the best-element property (see Kunimoto and Serrano (2010)) that implements a social choice function in rationalizable strategies but does not continuously implement. The example is available upon request.

<sup>28</sup>For comparison with Theorems 2 and 3, note that, if a SCF  $f$  is continuously implementable (under the assumptions made in Section 4) by a finite mechanism, then by Theorem 4, there is a finite mechanism that rationalizable implements  $f$ . A straightforward argument shows that  $f$  is partially implementable in pure Nash equilibrium. Hence,  $f$  is fully implementable in pure Nash equilibrium and so is Bayesian monotonic. Similarly, it can also be shown that  $f$  is rationalizable monotonic.

**Proposition 2** *A social choice function  $f$  is virtually continuously implementable by finite mechanisms if and only if it is virtually rationalizable implementable by finite mechanisms.*

While the formulations in Proposition 2 and Theorem 4 are similar, their implications are quite different. Indeed, Abreu and Matsushima (1992a)<sup>29</sup> showed that under complete information, and under some mild additional conditions, every social choice function is virtually rationalizable implementable with three or more players. Since mechanisms used in their paper are finite, we know by Proposition 2 that they also ensure continuous virtual implementation. Thus, under complete information, virtual continuous implementation is a lot less demanding than (exact) continuous implementation. However, we note that whether the characterization of virtual continuous implementation is weak in general is unclear. More precisely, in the Bayesian setting studied by Abreu and Matsushima (1992b), Bayesian Incentive Compatibility and a measurability condition are both necessary and sufficient for virtual implementation in rationalizable strategies and the strength of the measurability condition is controversial. In particular, Serrano and Vohra (2001) show that it may impose demanding restrictions.

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<sup>29</sup>In their setting, the (finite) set of outcomes is extended to the set of lotteries over outcomes and the natural metric is used over this set.

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## Appendix

### A Proof of Lemma 1

To show the lemma, we build a model  $\hat{\mathcal{T}}$  with  $\bar{\mathcal{T}} \subset \hat{\mathcal{T}}$  and denote by  $\sigma$  a strict continuous equilibrium in  $U(\mathcal{M}, \hat{\mathcal{T}})$ . Let  $\bar{\sigma}$  be  $\sigma|_{\bar{\mathcal{T}}}$  and recall that  $\bar{\sigma}$  must be a strict equilibrium in  $U(\mathcal{M}, \bar{\mathcal{T}})$ . The model  $\hat{\mathcal{T}}$  will be built in such a way that for any best-reply set  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  in  $U(\mathcal{M}, \bar{\mathcal{T}})$  satisfying  $\Sigma(\bar{\mathcal{T}}) \subset \bar{\sigma}(\bar{\mathcal{T}})$ , any  $i \in \mathcal{I}$ ,  $\bar{t}_i \in \bar{\mathcal{T}}_i$  and  $m_i \in \Sigma_i(\bar{t}_i)$ , there will exist a sequence of types  $\{\hat{t}_i[n]\}_{n=0}^\infty$  in  $\hat{\mathcal{T}}_i$  such that (1)  $\hat{t}_i[n] \rightarrow_P \bar{t}_i$  and (2)  $\sigma_i(\hat{t}_i[n]) = m_i$  for all  $n$ . This will yield  $(g \circ \sigma)(\hat{t}_i[n]) = g(m_i)$  for all  $n$  and so by the continuity requirement,  $g(m_i) = f(\bar{t}_i)$ . Thus, this will show that if a best-reply set  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  in  $U(\mathcal{M}, \bar{\mathcal{T}})$  satisfies  $\Sigma(\bar{\mathcal{T}}) \subset \bar{\sigma}(\bar{\mathcal{T}})$ , then  $g(\Sigma(\bar{t})) = \{f(\bar{t})\}$  for all  $\bar{t} \in \bar{\mathcal{T}}$ , as desired.

We define the set  $\mathcal{E}$  by:

$$\mathcal{E} := \cup_{q \in \mathbb{N}^*} \left\{ \frac{1}{q} \right\} \cup \{0\}.$$

In the sequel,  $C$  denotes the countable<sup>30</sup> set of strict Nash equilibria in  $U(\mathcal{M}, \bar{\mathcal{T}})$ . We will sometimes abuse notations and write  $\Sigma \subset \bar{\sigma}$  instead of  $\Sigma(\bar{\mathcal{T}}) \subset \bar{\sigma}(\bar{\mathcal{T}})$ . Now we build the model  $\hat{\mathcal{T}} = (\hat{\mathcal{T}}, \hat{\kappa})$  as follows. For each  $\varepsilon \in \mathcal{E}$ , equilibrium  $\bar{\sigma} \in C$ , best-reply set  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  in  $U(\mathcal{M}, \bar{\mathcal{T}})$  satisfying  $\Sigma(\bar{\mathcal{T}}) \subset \bar{\sigma}(\bar{\mathcal{T}})$ , each integer  $\ell$ , type  $\bar{t}_i \in \bar{\mathcal{T}}_i$  and message  $m_i \in \Sigma_i(\bar{t}_i)$ , we build inductively  $\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, \ell, \bar{t}_i, m_i]$  and set

$$\hat{\mathcal{T}}_i = \bigcup_{\varepsilon \in \mathcal{E}} \bigcup_{\bar{\sigma} \in C} \bigcup_{\Sigma: \Sigma \subset \bar{\sigma}} \bigcup_{\ell=1}^{\infty} \bigcup_{\bar{t}_i \in \bar{\mathcal{T}}_i} \bigcup_{m_i \in \Sigma_i(\bar{t}_i)} \hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, \ell, \bar{t}_i, m_i] \cup \bar{\mathcal{T}}_i.$$

Note that  $\hat{\mathcal{T}}_i$  is countable.<sup>31</sup>

For each  $\bar{\sigma} \in C$ , we know that for each best-reply set  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  in  $U(\mathcal{M}, \bar{\mathcal{T}})$  satisfying  $\Sigma(\bar{\mathcal{T}}) \subset \bar{\sigma}(\bar{\mathcal{T}})$ , player  $i$  of type  $\bar{t}_i \in \bar{\mathcal{T}}_i$  and  $m_i \in \Sigma_i(\bar{t}_i)$ , there exists  $\pi_{\bar{t}_i}^{m_i} \in \Delta(\Theta \times \bar{\mathcal{T}}_{-i} \times M_{-i})$  such that

$$\begin{aligned} \text{marg}_{\Theta \times \bar{\mathcal{T}}_{-i}} \pi_{\bar{t}_i}^{m_i} &= \bar{\kappa}(\bar{t}_i), \\ \text{marg}_{\bar{\mathcal{T}}_{-i} \times M_{-i}} \pi_{\bar{t}_i}^{m_i}(\bar{t}_{-i}, m_{-i}) &> 0 \Rightarrow m_{-i} \in \Sigma_{-i}(\bar{t}_{-i}), \end{aligned}$$

and

$$m_i \in BR_i(\text{marg}_{\Theta \times M_{-i}} \pi_{\bar{t}_i}^{m_i} \mid \mathcal{M}).$$

In addition, since each  $\bar{\sigma} \in C$  is a strict equilibrium in  $U(\mathcal{M}, \bar{\mathcal{T}})$ , we know that for each player  $i$  and message  $m_i \in \Sigma_i(\bar{\mathcal{T}}_i) \subset \bar{\sigma}_i(\bar{\mathcal{T}}_i)$ , there exists  $\pi_*^{m_i} \in \Delta(\Theta \times \bar{\mathcal{T}}_{-i} \times M_{-i})$  such

<sup>30</sup>Recall that  $M$  is countable and  $\bar{\mathcal{T}}$  is finite.

<sup>31</sup>Indeed, the set of best-reply sets  $\Sigma$  satisfying  $\Sigma \subset \bar{\sigma}$  is finite.

that  $\{m_i\} = BR_i(\text{marg}_{\Theta \times M_{-i}} \pi_*^{m_i} \mid \mathcal{M})$ . We note that for all  $\varepsilon > 0$  :

$$\{m_i\} = BR_i((1 - \varepsilon) \text{marg}_{\Theta \times M_{-i}} \pi_{\bar{t}_i}^{m_i} + \varepsilon \text{marg}_{\Theta \times M_{-i}} \pi_*^{m_i} \mid \mathcal{M}). \quad (3)$$

For each equilibrium  $\bar{\sigma} \in C$ , best-reply set  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  in  $U(\mathcal{M}, \bar{T})$  satisfying  $\Sigma(\bar{T}) \subset \bar{\sigma}(\bar{T})$ , player  $i \in \mathcal{I}$  and message  $m_i \in \Sigma_i(\bar{T}_i)$ , we let  $t_i[\bar{\sigma}, \Sigma, m_i] \in \bar{T}_i$  be a type such that  $\bar{\sigma}_i(t_i[\bar{\sigma}, \Sigma, m_i]) = m_i$ , which is well-defined since  $m_i \in \Sigma_i(\bar{T}_i) \subset \bar{\sigma}_i(\bar{T}_i)$ .

We now define  $\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, \ell, \bar{t}_i, m_i]$  inductively on  $\ell$ . First, we let  $\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, 1, \bar{t}_i, m_i]$  be such that  $\hat{\kappa}(\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, 1, \bar{t}_i, m_i])$  satisfies

$$\hat{\kappa}(\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, 1, \bar{t}_i, m_i]) = (1 - \varepsilon) \pi_{\bar{t}_i}^{m_i} \circ \left( \tau_{-i}^{\varepsilon, 1} \right)^{-1} + \varepsilon \pi_*^{m_i} \circ \left( \tau_{-i}^{\varepsilon, 1} \right)^{-1} \quad (4)$$

where  $\left( \tau_{-i}^{\varepsilon, 1} \right)^{-1}$  stands for the preimage of the function  $\tau_{-i}^{\varepsilon, 1} : \Theta \times \bar{T}_{-i} \times M_{-i} \rightarrow \Theta \times \hat{T}_{-i}$ , defined by  $\tau_{-i}^{\varepsilon, 1}(\theta, \bar{t}_{-i}, m_{-i}) = (\theta, t_{-i}[\bar{\sigma}, \Sigma, m_{-i}])$ . Note that  $\left( \tau_{-i}^{\varepsilon, 1} \right)^{-1}$  is a set-valued function and that  $\left( \tau_{-i}^{\varepsilon, 1} \right)^{-1}(\theta, t_{-i}) = \{\emptyset\}$  whenever  $t_{-i} \notin \bar{T}_{-i}$ . Now for each  $\ell \geq 2$ , define  $\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, \ell, \bar{t}_i, m_i]$  inductively by

$$\hat{\kappa}(\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, \ell, \bar{t}_i, m_i]) = (1 - \varepsilon) \pi_{\bar{t}_i}^{m_i} \circ \left( \tau_{-i}^{\varepsilon, \ell} \right)^{-1} + \varepsilon \pi_*^{m_i} \circ \left( \tau_{-i}^{\varepsilon, \ell} \right)^{-1}$$

where  $\left( \tau_{-i}^{\varepsilon, \ell} \right)^{-1}$  stands for the preimage of the function  $\tau_{-i}^{\varepsilon, \ell} : \Theta \times \bar{T}_{-i} \times M_{-i} \rightarrow \Theta \times \hat{T}_{-i}$ , defined by  $\tau_{-i}^{\varepsilon, \ell}(\theta, \bar{t}_{-i}, m_{-i}) = (\theta, \hat{t}_{-i}[\varepsilon, \bar{\sigma}, \Sigma, \ell - 1, \bar{t}_{-i}, m_{-i}])$ .

**Claim 1** For each  $\bar{\sigma} \in C$ , best-reply set  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  in  $U(\mathcal{M}, \bar{T})$  satisfying  $\Sigma(\bar{T}) \subset \bar{\sigma}(\bar{T})$ ,  $i, \bar{t}_i \in \bar{T}_i$  and  $m_i \in \Sigma_i(\bar{t}_i) : \hat{t}_i[\hat{\varepsilon}(\ell), \bar{\sigma}, \Sigma, \ell, \bar{t}_i, m_i] \rightarrow_P \bar{t}_i$  as  $\ell \rightarrow \infty$  for some mapping  $\hat{\varepsilon}$  taking values in  $\mathcal{E} \setminus \{0\}$ .

To prove this claim, we will use the following well-known lemma.

**Lemma 2 (Mertens and Zamir (1985) and Brandenburger and Dekel (1993))**

Let  $\mathcal{T} = (T, \kappa)$  be any model such that  $\Theta \times T$  is complete and separable and  $\kappa(\cdot)$  is a continuous function of  $t_i$  for each player  $i$ . Then, the mapping  $h : T \rightarrow T^*$  is continuous.

**Proof of Claim 1.** In the sequel, we will note  $\bar{h}$  the mapping that projects  $\bar{T}$  into  $T^*$  and, in a similar way,  $\hat{h}$  the mapping from  $\hat{T}$  to  $T^*$ . By Lemma 2, these mappings are continuous<sup>32</sup>.

<sup>32</sup> A type in  $\hat{T}_i$  is either in  $\bar{T}_i$  – which is endowed with the discrete topology, say  $\tau_{T_i}$  – or in  $\hat{T}_i \setminus \bar{T}_i$ . Any point in  $\hat{T}_i \setminus \bar{T}_i$  is identified with an element of the set  $\mathcal{E} \times C \times \{\Sigma : \bar{T} \rightarrow 2^M \setminus \{\emptyset\} \mid \Sigma(\bar{T}) \in \mathcal{S}\} \times \mathbb{N} \times \bar{T}_i \times M_i$  where  $\mathcal{S}$  is defined as the set of profile of messages  $M'$  such that  $M' \subset \bar{\sigma}(\bar{T})$  for some  $\bar{\sigma}$  in  $C$ . The sets  $C, \{\Sigma : \bar{T} \rightarrow M \mid \Sigma(\bar{T}) \in \mathcal{S}\}, \mathbb{N}, \bar{T}_i$  and  $M_i$  are all (countable and) endowed with the discrete topology while  $\mathcal{E}$  is endowed with the usual topology on  $\mathbb{R}$  induced on  $\mathcal{E}$ . Finally,  $\mathcal{E} \times C \times \{\Sigma : \bar{T} \rightarrow M \mid \Sigma(\bar{T}) \in \mathcal{S}\} \times \mathbb{N} \times \bar{T}_i \times M_i$  is endowed with the product topology; call this topology  $\tau_{\hat{T}_i \setminus \bar{T}_i}$ . The topology over  $\hat{T}_i$  is the coarsest topology that contains  $\tau_{\bar{T}_i} \cup \tau_{\hat{T}_i \setminus \bar{T}_i}$ . It can easily be checked that under such a topology,  $\hat{T}$  satisfies the conditions of Lemma 2.

In the sequel, we fix  $\bar{\sigma}$  and a best-reply set  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  in  $U(\mathcal{M}, \bar{T})$  satisfying  $\Sigma(\bar{T}) \subset \bar{\sigma}(\bar{T})$ . Since for all  $\ell' \geq 1$  and all  $\bar{t}_i, m_i : \hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i] \rightarrow \hat{t}_i[0, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i]$  as  $\varepsilon \rightarrow 0$ , by continuity, for all  $\ell' \geq 1$ , all  $\ell \geq 1$  and all  $\bar{t}_i, m_i : \hat{h}_i^\ell(\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i]) \rightarrow \hat{h}_i^\ell(\hat{t}_i[0, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i])$  as  $\varepsilon \rightarrow 0$ .

Let us now show inductively on  $\ell$  that for all  $\ell \geq 1$  and  $\ell' \geq \ell : \hat{h}_i^\ell(\hat{t}_i[0, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i]) = \bar{h}_i^\ell(\bar{t}_i)$  for each  $i, \bar{t}_i$  and  $m_i \in \Sigma_i(\bar{t}_i)$ . First notice that the first order beliefs are equal, i.e., for all  $\ell' \geq 1$  :

$$\begin{aligned} \hat{h}_i^1(\hat{t}_i[0, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i]) &= \text{marg}_\Theta \hat{\kappa}(\hat{t}_i[0, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i]) \\ &= \text{marg}_\Theta \pi_{\bar{t}_i}^{m_i} \circ \left( \tau_{-i}^{0, \ell'} \right)^{-1} \\ &= \text{marg}_\Theta \pi_{\bar{t}_i}^{m_i} = \text{marg}_\Theta \bar{\kappa}(\bar{t}_i) = \bar{h}_i^1(\bar{t}_i) \end{aligned}$$

where the third and the fourth equalities are by definition of  $\tau_{-i}^{0, \ell'}$  and  $\pi_{\bar{t}_i}^{m_i}$  respectively. Now fix some  $\ell \geq 2$  and let  $L$  be the set of all belief profiles of players other than  $i$  at order  $\ell - 1$ . Toward an induction, assume that for all  $\ell' \geq \ell - 1 : \hat{h}_j^{\ell-1}(\hat{t}_j[0, \bar{\sigma}, \Sigma, \ell', \bar{t}_j, m_j]) = \bar{h}_j^{\ell-1}(\bar{t}_j)$  for each  $j, \bar{t}_j \in \bar{T}_j$  and  $m_j \in \Sigma_j(\bar{t}_j)$ . Then for all  $\ell' \geq \ell : \text{proj}_{\Theta \times L} \circ \left( id_\Theta \times \hat{h}_{-i} \right) \circ \tau_{-i}^{0, \ell'} = \overline{\text{proj}}_{\Theta \times L} \circ \left( id_\Theta \times \bar{h}_{-i} \times id_{M_{-i}} \right)$  where  $id_\Theta$  (resp.  $id_{M_{-i}}$ ) is the identity mapping from  $\Theta$  to  $\Theta$  (resp. from  $M_{-i}$  to  $M_{-i}$ ) while  $\text{proj}_{\Theta \times L}$  (resp.  $\overline{\text{proj}}_{\Theta \times L}$ ) is the projection mapping from  $\Theta \times T_{-i}^*$  to  $\Theta \times L$  (resp. from  $\Theta \times T_{-i}^* \times M_{-i}$  to  $\Theta \times L$ ); hence, for all  $\ell' \geq \ell$  :

$$\begin{aligned} \text{marg}_{\Theta \times L} \hat{\kappa}(\hat{t}_i[0, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i]) \circ \left( id_\Theta \times \hat{h}_{-i} \right)^{-1} &= \text{marg}_{\Theta \times L} \pi_{\bar{t}_i}^{m_i} \circ \left( \tau_{-i}^{0, \ell'} \right)^{-1} \circ \left( id_\Theta \times \hat{h}_{-i} \right)^{-1} \\ &= \pi_{\bar{t}_i}^{m_i} \circ \left( \tau_{-i}^{0, \ell'} \right)^{-1} \circ \left( id_\Theta \times \hat{h}_{-i} \right)^{-1} \circ \left( \text{proj}_{\Theta \times L} \right)^{-1} \\ &= \pi_{\bar{t}_i}^{m_i} \circ \left( id_\Theta \times \bar{h}_{-i} \times id_{M_{-i}} \right)^{-1} \circ \left( \overline{\text{proj}}_{\Theta \times L} \right)^{-1} \\ &= \text{marg}_{\Theta \times L} \pi_{\bar{t}_i}^{m_i} \circ \left( id_\Theta \times \bar{h}_{-i} \times id_{M_{-i}} \right)^{-1} \\ &= \text{marg}_{\Theta \times L} \bar{\kappa}(\bar{t}_i) \circ \left( id_\Theta \times \bar{h}_{-i} \right)^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{h}_i^\ell(\hat{t}_i[0, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i]) &= \delta_{\hat{h}_i^{\ell-1}(\hat{t}_i[0, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i])} \times \text{marg}_{\Theta \times L} \hat{\kappa}(\hat{t}_i[0, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i]) \circ \left( id_\Theta \times \hat{h}_{-i} \right)^{-1} \\ &= \delta_{\bar{h}_i^{\ell-1}(\bar{t}_i)} \times \text{marg}_{\Theta \times L} \bar{\kappa}(\bar{t}_i) \circ \left( id_\Theta \times \bar{h}_{-i} \right)^{-1} = \bar{h}_i^\ell(\bar{t}_i) \end{aligned}$$

showing that  $\hat{h}_i^\ell(\hat{t}_i[0, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i]) = \bar{h}_i^\ell(\bar{t}_i)$ . Thus, we have proved that for all  $\ell \geq 1$  and  $\ell' \geq \ell : \hat{h}_i^\ell(\hat{t}_i[0, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i]) = \bar{h}_i^\ell(\bar{t}_i)$  for each  $i, \bar{t}_i$  and  $m_i \in \Sigma_i(\bar{t}_i)$  which means that  $\hat{t}_i[0, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i] \rightarrow_P \bar{t}_i$  as  $\ell' \rightarrow \infty$  for each  $i, \bar{t}_i$  and  $m_i \in \Sigma_i(\bar{t}_i)$ . In addition, we know that for all  $\ell' \geq 1$ , all  $i, \bar{t}_i, m_i : \hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i] \rightarrow_P \hat{t}_i[0, \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i]$  as  $\varepsilon \rightarrow 0$ . Since  $T^*$  is a metrizable space,  $\hat{t}_i[\hat{\varepsilon}(\ell'), \bar{\sigma}, \Sigma, \ell', \bar{t}_i, m_i] \rightarrow_P \bar{t}_i$  as  $\ell' \rightarrow \infty$  for some function  $\hat{\varepsilon} : \mathbb{N}^* \rightarrow \mathcal{E} \setminus \{0\}$  satisfying  $\lim_{\ell' \rightarrow \infty} \hat{\varepsilon}(\ell') = 0$ . ■

**Claim 2** For each  $\bar{\sigma} \in C$ , each best-reply set  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  in  $U(\mathcal{M}, \bar{T})$  satisfying  $\Sigma(\bar{T}) \subset \bar{\sigma}(\bar{T})$ ,  $\varepsilon \in \mathcal{E} \setminus \{0\}$ ,  $\ell, i, \bar{t}_i \in \bar{T}_i$  and  $m_i \in \Sigma_i(\bar{t}_i)$ , we have:  $\sigma_i(\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, \ell, \bar{t}_i, m_i]) = m_i$  for any equilibrium  $\sigma$  of  $U(\mathcal{M}, \hat{T})$  satisfying  $\sigma|_{\hat{T}} = \bar{\sigma}$ .

**Proof.** Fix an equilibrium  $\sigma$  of  $U(\mathcal{M}, \hat{T})$  satisfying  $\sigma|_{\hat{T}} = \bar{\sigma}$ . We will show by induction on  $\ell$  that for all  $\varepsilon > 0$  and  $\ell \geq 1$ :  $\sigma_i(\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, \ell, \bar{t}_i, m_i]) = m_i$  for all player  $i$ , type  $\bar{t}_i \in \bar{T}_i$  and all messages  $m_i \in \Sigma_i(\bar{t}_i)$ .

Recall that, by construction, for all  $\bar{t}_i \in \bar{T}_i$  and  $m_i \in \Sigma_i(\bar{t}_i)$ :  $t_i[\bar{\sigma}, \Sigma, m_i] \in \bar{T}_i$  is the type such that  $\sigma_i(t_i[\bar{\sigma}, \Sigma, m_i]) = \bar{\sigma}_i(t_i[\bar{\sigma}, \Sigma, m_i]) = m_i$ . First, fix  $\varepsilon > 0$ ,  $\bar{t}_i \in \bar{T}_i$  and  $m_i \in \Sigma_i(\bar{t}_i)$  and let us prove that  $\sigma_i(\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, 1, \bar{t}_i, m_i]) = m_i$ . For each  $\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, 1, \bar{t}_i, m_i]$ , define the belief

$$\pi_i^{\varepsilon, 1} = \hat{\kappa}(\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, 1, \bar{t}_i, m_i]) \circ \gamma^{-1} \in \Delta(\Theta \times \hat{T}_{-i} \times M_{-i})$$

where  $\gamma : (\theta, t_{-i}[\bar{\sigma}, \Sigma, m_{-i}]) \mapsto (\theta, t_{-i}[\bar{\sigma}, \Sigma, m_{-i}], m_{-i})$ . Note that by construction,  $\pi_i^{\varepsilon, 1}$  is the belief of type  $\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, 1, \bar{t}_i, m_i]$  on  $\Theta \times \hat{T}_{-i} \times M_{-i}$  when he believes that  $m_{-i}$  is played at each  $(\theta, t_{-i}[\bar{\sigma}, \Sigma, m_{-i}])$ . Hence, for each  $\varepsilon \geq 0$ ,  $\pi_i^{\varepsilon, 1}$  corresponds to beliefs of type  $\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, 1, \bar{t}_i, m_i]$  when the equilibrium  $\sigma$  is played. Now, by Equation (4), the belief  $\pi_i^{\varepsilon, 1}$  of type  $\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, 1, \bar{t}_i, m_i]$  satisfies

$$\begin{aligned} & \text{marg}_{\Theta \times M_{-i}} \pi_i^{\varepsilon, 1} \\ &= (1 - \varepsilon) \text{marg}_{\Theta \times M_{-i}} \pi_{\bar{t}_i}^{m_i} \circ \left(\tau_{-i}^{\varepsilon, 1}\right)^{-1} \circ (\gamma)^{-1} + \varepsilon \text{marg}_{\Theta \times M_{-i}} \pi_*^{m_i} \circ \left(\tau_{-i}^{\varepsilon, 1}\right)^{-1} \circ (\gamma)^{-1} \\ &= (1 - \varepsilon) \text{marg}_{\Theta \times M_{-i}} \pi_{\bar{t}_i}^{m_i} + \varepsilon \text{marg}_{\Theta \times M_{-i}} \pi_*^{m_i}. \end{aligned}$$

For  $\varepsilon > 0$ , by Equation (3) and the construction of  $\pi_i^{\varepsilon, 1}$ ,  $\{m_i\} = BR_i(\text{marg}_{\Theta \times M_{-i}} \pi_i^{\varepsilon, 1} | \mathcal{M})$  and  $\sigma_i(\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, 1, \bar{t}_i, m_i]) = m_i$ .

Now, for each  $\ell \geq 2$ , proceed by induction and assume that  $\sigma_{-i}(\hat{t}_{-i}[\varepsilon, \bar{\sigma}, \Sigma, \ell - 1, \bar{t}_{-i}, m_{-i}]) = m_{-i}$  for any  $\bar{t}_{-i} \in \bar{T}_{-i}$ ,  $m_{-i} \in \Sigma_{-i}(\bar{t}_{-i})$  and  $\varepsilon > 0$ . Fix  $\varepsilon > 0$ ,  $i, \bar{t}_i \in \bar{T}_i$  and  $m_i \in \Sigma_i(\bar{t}_i)$ . For each  $\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, \ell, \bar{t}_i, m_i]$ , define the belief

$$\pi_i^{\varepsilon, \ell} = \hat{\kappa}(\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, \ell, \bar{t}_i, m_i]) \circ \gamma_\ell^{-1} \in \Delta(\Theta \times \hat{T}_{-i} \times M_{-i})$$

where  $\gamma_\ell : (\theta, \hat{t}_{-i}[\varepsilon, \bar{\sigma}, \Sigma, \ell - 1, \bar{t}_{-i}, m_{-i}]) \mapsto (\theta, \hat{t}_{-i}[\varepsilon, \bar{\sigma}, \Sigma, \ell - 1, \bar{t}_{-i}, m_{-i}], m_{-i})$ .

Note that, by construction,  $\pi_i^{\varepsilon, \ell}$  is the belief of type  $\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, \ell, \bar{t}_i, m_i]$  on  $\Theta \times \hat{T}_{-i} \times M_{-i}$  when he believes that  $m_{-i}$  is played at each  $(\theta, \hat{t}_{-i}[\varepsilon, \bar{\sigma}, \Sigma, \ell - 1, \bar{t}_{-i}, m_{-i}])$ . Hence, by the induction hypothesis, for each  $\varepsilon \geq 0$ ,  $\pi_i^{\varepsilon, \ell}$  corresponds to beliefs of type  $\hat{t}_i[\varepsilon, \bar{\sigma}, \Sigma, \ell, \bar{t}_i, m_i]$  when the equilibrium  $\sigma$  is played. The end of the proof mimics the case  $\ell = 1$ . ■

## B Proof of Theorem 3

Suppose  $f$  is strictly continuously implementable on  $(\bar{T}, \bar{\kappa})$  and fix a mechanism  $\mathcal{M}$  that strictly continuously implements  $f$ . Take the strict equilibrium  $\bar{\sigma} : \bar{T} \rightarrow M$  obtained in Lemma 1 and pick any unacceptable deception  $\beta : \bar{T} \rightarrow \bar{T}$ . Note that there must exist some  $\bar{t}$  and  $\bar{t}'$  in  $\bar{T}$  such that  $\bar{t}' \in \beta(\bar{t})$  with  $f(\bar{t}') \neq f(\bar{t})$ . Given that  $g \circ \bar{\sigma} = f$ , this yields  $(g \circ \bar{\sigma} \circ \beta)(\bar{t}) \neq \{f(\bar{t})\}$ . Thus, Lemma 1 together with the fact that the correspondence  $\bar{\sigma} \circ \beta = \times_{i \in \mathcal{I}} (\bar{\sigma}_i \circ \beta_i)$  satisfies  $(\bar{\sigma} \circ \beta)(\bar{T}) \subset \bar{\sigma}(\bar{T})$ , implies that  $\bar{\sigma} \circ \beta$  cannot be a best-reply set. In other words, there exist  $i, t_i, t'_i \in \beta_i(t_i)$  and  $m_i^* = \bar{\sigma}_i(t'_i) \in \bar{\sigma}_i \circ \beta_i(t_i)$  such that for all  $\nu_i \in \Delta(\Theta \times \bar{T}_{-i} \times M_{-i})$  satisfying [i]  $\nu_i(\theta, t_{-i}, m_{-i}) > 0 \Rightarrow m_j \in (\bar{\sigma}_j \circ \beta_j)(t_j)$  for all  $j \neq i$ , [ii]  $\sum_{m_{-i}} \nu_i(\theta, t_{-i}, m_{-i}) = \bar{\kappa}(t_i)[\theta, t_{-i}]$  for each  $(\theta, t_{-i})$ , there exists  $\tilde{m}_i \neq m_i^*$  such that

$$\sum_{\theta, m_{-i}} \text{marg}_{\Theta \times M_{-i}} \nu_i(\theta, m_{-i}) u_i(g(\tilde{m}_i, m_{-i}), \theta) > \sum_{\theta, m_{-i}} \text{marg}_{\Theta \times M_{-i}} \nu_i(\theta, m_{-i}) u_i(g(m_i^*, m_{-i}), \theta). \quad (5)$$

Now, let  $\psi_i \in \Delta(\Theta \times \bar{T}_{-i} \times \bar{T}_{-i})$  be such that

$$\begin{aligned} [1] \quad & \psi_i(\theta, t_{-i}, t'_{-i}) > 0 \Rightarrow t'_j \in \beta_j(t_j) \text{ for each } j \neq i, \text{ and,} \\ [2] \quad & \sum_{t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) = \bar{\kappa}(t_i)[\theta, t_{-i}] \text{ for each } (\theta, t_{-i}). \end{aligned}$$

Set  $\nu_i(\theta, t_{-i}, m_{-i}) := \sum_{t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) \bar{\sigma}_{-i}(m_{-i} \mid t'_{-i})$ . Note that  $\nu_i$  satisfies [i] and [ii].

Thus, there exists  $\tilde{m}_i \neq m_i^*$  such that Equation (5) is satisfied. Or equivalently,

$$\begin{aligned} & \sum_{\theta, m_{-i}} \sum_{t_{-i}} \sum_{t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) \bar{\sigma}_{-i}(m_{-i} \mid t'_{-i}) u_i(g(\tilde{m}_i, m_{-i}), \theta) \\ > & \sum_{\theta, m_{-i}} \sum_{t_{-i}} \sum_{t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) \bar{\sigma}_{-i}(m_{-i} \mid t'_{-i}) u_i(g(m_i^*, m_{-i}), \theta), \end{aligned}$$

and so,

$$\begin{aligned} & \sum_{\theta, t_{-i}, t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) u_i(g(\tilde{m}_i, \bar{\sigma}_{-i}(t'_{-i})), \theta) \\ > & \sum_{\theta, t_{-i}, t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) u_i(g(m_i^*, \bar{\sigma}_{-i}(t'_{-i})), \theta), \end{aligned}$$

which yields

$$\sum_{\theta, t_{-i}, t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) u_i(g^*(t'_{-i}), \theta) > \sum_{\theta, t_{-i}, t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) u_i(f(t'_i, t'_{-i}), \theta)$$

where  $y^*(\cdot) := g(\tilde{m}_i, \bar{\sigma}_{-i}(\cdot))$  and using the fact that  $m_i^* = \bar{\sigma}_i(t'_i)$  and  $g \circ \bar{\sigma} = f$ . Now to complete the proof, we have to show that for each  $\tilde{t}_i \in \bar{T}_i$ :

$$\sum_{\theta, t_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, t_{-i}] u_i(f(\tilde{t}_i, t_{-i}), \theta) \geq \sum_{\theta, t_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, t_{-i}] u_i(y^*(t_{-i}), \theta)$$

This is true because for each  $\tilde{t}_i \in \bar{T}_i$ :

$$\begin{aligned} & \sum_{\theta, t_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, t_{-i}] u_i(y^*(t_{-i}), \theta) \\ = & \sum_{\theta, t_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, t_{-i}] u_i(g(\tilde{m}_i, \bar{\sigma}_{-i}(t_{-i})), \theta) \\ \leq & \sum_{\theta, t_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, t_{-i}] u_i(g(\bar{\sigma}_i(\tilde{t}_i), \bar{\sigma}_{-i}(t_{-i})), \theta) \\ = & \sum_{\theta, t_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, t_{-i}] u_i(f(\tilde{t}_i, t_{-i}), \theta) \end{aligned}$$

where the inequality above comes from the fact that  $\bar{\sigma}$  is an equilibrium in  $U(\mathcal{M}, \bar{\mathcal{T}})$  while the last equality is obtained using the fact  $g \circ \bar{\sigma} = f$ . Note that since  $\bar{\sigma}$  is a strict equilibrium and  $\tilde{m}_i \neq m_i^* = \bar{\sigma}_i(t'_i)$ , the above inequality is strict when  $\tilde{t}_i = t'_i$ . This proves that  $f$  is strict interim rationalizable monotonic.

## C Proof of Proposition 1

Before proving this proposition, let us make the following simple remark.

**Remark 1** *It is easily checked that Assumption 1 yields that, for all  $i$ , there exists  $\hat{y}_i \in \Delta(A)$  such that the following holds. For any  $\phi_i \in \Delta(\Theta)$ , there exists  $y \in \Delta(A)$  such that*

$$\sum_{\theta} \phi_i(\theta) u_i(y, \theta) > \sum_{\theta} \phi_i(\theta) u_i(\hat{y}_i, \theta).$$

*Note that here  $\hat{y}_i$  and  $y$  are lotteries (and not mappings).*

Before proving the proposition, we will need the following lemma which is an extension to the Bayesian setting of Lemma 1 in Bergemann and Morris (2011).<sup>33</sup>

**Lemma 3** *If  $f$  is interim rationalizable monotonic then  $f$  is incentive compatible, i.e., for all  $i$  and  $t_i \in \bar{T}_i$ :*

$$\sum_{\theta, t_{-i}} \bar{\kappa}(t_i)[\theta, t_{-i}] u_i(f(t_i, t_{-i}), \theta) \geq \sum_{\theta, t_{-i}} \bar{\kappa}(t_i)[\theta, t_{-i}] u_i(f(t'_i, t_{-i}), \theta)$$

*for all  $t'_i$ .*

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<sup>33</sup>It is easily checked that as in Bergemann and Morris (2011), we actually prove here that  $f$  is semi-strict incentive compatible, i.e. a strengthening of incentive compatibility.

**Proof.** Assume  $f$  is interim rationalizable monotonic. Fix any player  $i$  and  $t_i, t'_i$ . If  $f(t_i, t_{-i}) = f(t'_i, t_{-i})$  for all  $t_{-i}$ , we trivially have

$$\sum_{\theta, t_{-i}} \bar{\kappa}(t_i)[\theta, t_{-i}] u_i(f(t_i, t_{-i}), \theta) \geq \sum_{\theta, t_{-i}} \bar{\kappa}(t_i)[\theta, t_{-i}] u_i(f(t'_i, t_{-i}), \theta).$$

Now, assume that  $f(t_i, \hat{t}_{-i}) \neq f(t'_i, \hat{t}_{-i})$  for some  $\hat{t}_{-i}$ . To complete the proof, we will show that

$$\sum_{\theta, t_{-i}} \bar{\kappa}(t_i)[\theta, t_{-i}] u_i(f(t_i, t_{-i}), \theta) > \sum_{\theta, t_{-i}} \bar{\kappa}(t_i)[\theta, t_{-i}] u_i(f(t'_i, t_{-i}), \theta).$$

Build the deception  $\beta$  as follows: for each  $j \neq i$ ,  $\beta_j(t_j) = \{t_j\}$  and  $\beta_i(\tilde{t}_i) = \begin{cases} \{t_i, t'_i\} & \text{if } \tilde{t}_i = t_i \\ \{\tilde{t}_i\} & \text{otherwise} \end{cases}$ .

Note that  $\beta$  is an unacceptable deception: indeed,  $(t'_i, \hat{t}_{-i}) \in \beta(t_i, \hat{t}_{-i})$  while  $f(t'_i, \hat{t}_{-i}) \neq f(t_i, \hat{t}_{-i})$ . It is easily checked that, by interim rationalizable monotonicity, the following condition holds for  $t_i$  and  $t'_i \in \beta_i(t_i)$ . For every  $\psi_i \in \Delta(\Theta \times \bar{T}_{-i} \times \bar{T}_{-i})$  such that

$$[1] \quad \psi_i(\theta, t_{-i}, t'_{-i}) > 0 \Rightarrow t'_j \in \beta_j(t_j) \text{ for each } j \neq i$$

$$[2] \quad \sum_{t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) = \bar{\kappa}(t_i)[\theta, t_{-i}] \text{ for each } (\theta, t_{-i})$$

there exists  $y^* : \bar{T}_{-i} \rightarrow A$  such that

$$\sum_{\theta, t_{-i}, t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) u_i(y^*(t'_{-i}), \theta) > \sum_{\theta, t_{-i}, t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) u_i(f(t'_i, t'_{-i}), \theta)$$

and

$$\sum_{\theta, t_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, t_{-i}] u_i(f(\tilde{t}_i, t_{-i}), \theta) \geq \sum_{\theta, t_{-i}} \bar{\kappa}(\tilde{t}_i)[\theta, t_{-i}] u_i(y^*(t_{-i}), \theta)$$

for each  $\tilde{t}_i \in \bar{T}_i$ .

Now to complete the proof, build  $\psi_i(\theta, t_{-i}, t'_{-i}) = \bar{\kappa}(t_i)[\theta, t_{-i}] \delta_{\{t_{-i}\}}(t'_{-i})$  where  $\delta_{\{t_{-i}\}}$  is the dirac measure on  $\{t_{-i}\}$ ; it is clear that [1]  $\psi_i(\theta, t_{-i}, t'_{-i}) > 0 \Rightarrow t'_j \in \beta_j(t_j)$  for each  $j \neq i$  and that [2]  $\sum_{t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) = \bar{\kappa}(t_i)[\theta, t_{-i}]$  for each  $(\theta, t_{-i})$ . Hence, there must exist  $y^* : \bar{T}_{-i} \rightarrow \Delta(A)$  such that

$$\sum_{\theta, t_{-i}} \bar{\kappa}(t_i)[\theta, t_{-i}] u_i(y^*(t_{-i}), \theta) > \sum_{\theta, t_{-i}} \bar{\kappa}(t_i)[\theta, t_{-i}] u_i(f(t'_i, t_{-i}), \theta) \quad (6)$$

and

$$\sum_{\theta, t_{-i}} \bar{\kappa}(t_i)[\theta, t_{-i}] u_i(f(t_i, t_{-i}), \theta) \geq \sum_{\theta, t_{-i}} \bar{\kappa}(t_i)[\theta, t_{-i}] u_i(y^*(t_{-i}), \theta). \quad (7)$$

Equations (6) and (7) above yield

$$\sum_{\theta, t_{-i}} \bar{\kappa}(t_i)[\theta, t_{-i}] u_i(f(t_i, t_{-i}), \theta) > \sum_{\theta, t_{-i}} \bar{\kappa}(t_i)[\theta, t_{-i}] u_i(f(t'_i, t_{-i}), \theta),$$

the desired result. ■

We are now in a position to prove Proposition 1.

**Proof of Proposition 1.** The proof is by construction of a canonical mechanism  $\mathcal{M}$  that implements in rationalizable strategies  $f$  when  $I \geq 3$ . Each agent  $i$  sends a message  $m_i = (m_i^1, m_i^2, m_i^3, m_i^4)$ , where  $m_i^1 \in \bar{T}_i$ ,  $m_i^2 \in \mathbb{N}$ ,  $m_i^3(\cdot) \in Y_i$  and  $m_i^4 \in \Delta(A)$ . The outcome  $g(m)$  is determined by the following rules:

*Rule 1:* If  $m_i^2 = 1$  for all  $i$ , pick  $f(m^1)$

*Rule 2:* If there exists  $j$  such that  $m_i^2 = 1$  for all  $i \neq j$  and  $m_j^2 > 1$ , then pick  $m_j^3(m_{-j}^1)$  with probability  $1 - \frac{1}{m_j^2+1}$  and  $\bar{y}_j(m_{-j}^1)$  with probability  $\frac{1}{m_j^2+1}$  where  $\bar{y}_j(\cdot)$  is as defined in Assumption 1.

*Rule 3:* In all other cases, for each  $i$ , with probability  $\frac{1}{I} \left(1 - \frac{1}{m_i^2+1}\right)$  pick  $m_i^4$  and with probability  $\frac{1}{I} \left(\frac{1}{m_i^2+1}\right)$  pick  $\hat{y}_i$  where  $\hat{y}_i$  is defined in Remark 1.

The following three claims complete the proof.

**Claim 3** For each player  $i$  and type  $t_i : R_i(t_i | \mathcal{M}, \bar{T})$  is non-empty.

**Proof.** We will check that for any  $m_i^3$  and  $m_i^4$ ,  $(t_i, 1, m_i^3, m_i^4) \in R_i(t_i | \mathcal{M}, \bar{T})$ . To be more precise, for each  $t_i$ , we define  $\Sigma_i(t_i) = \{m_i \in M_i \mid m_i^1 = t_i \text{ and } m_i^2 = 1\}$  and check that  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  is a best-reply set in  $U(\mathcal{M}, \bar{T})$ . Fix  $i$ ,  $t_i$  and any selection  $m_{-i}(\cdot)$  of  $\Sigma_{-i}(\cdot)$ . For  $\pi_i \in \Delta(\Theta \times \bar{T}_{-i} \times M_{-i})$  defined by  $\pi_i(\theta, t_{-i}, m_{-i}) = \bar{\kappa}(t_i)[\theta, t_{-i}] \delta_{m_{-i}(t_{-i})}(m_{-i})$ , we clearly have  $\text{marg}_{\Theta \times \bar{T}_{-i}} \pi_i = \bar{\kappa}(t_i)$  and  $\pi_i(\theta, t_{-i}, m_{-i}) > 0 \implies m_{-i} \in \Sigma_{-i}(t_{-i})$ . In addition, it is easily checked that for all  $m_i \in \Sigma_i(t_i)$ :

$$\sum_{\theta, m_{-i}} \text{marg}_{\Theta \times M_{-i}} \pi_i(\theta, m_{-i}) u_i(g(m_i, m_{-i}), \theta) = \sum_{\theta, t_{-i}} \bar{\kappa}(t_i)[\theta, t_{-i}] u_i(f(t_i, t_{-i}), \theta).$$

Since, by Lemma 3,  $f$  is incentive compatible and since  $m_i^3(\cdot) \in Y_i$ , by construction of the mechanism, we have  $m_i \in BR_i(\text{marg}_{\Theta \times M_{-i}} \pi_i | \mathcal{M})$  for all  $m_i \in \Sigma_i(t_i)$  proving that  $\Sigma = (\Sigma_i)_{i \in \mathcal{I}}$  is indeed a best-reply set in  $U(\mathcal{M}, \bar{T})$ . ■

So far we have proved that the set of rationalizable strategy profiles in  $U(\mathcal{M}, \bar{T})$  is non-empty. It remains to show that all rationalizable strategy profiles yield the desired outcome at any profile of types. This is established via the following two claims.

**Claim 4** For any player  $i$  and type  $t_i$ , it is never a best-reply to send a message with  $m_i^2 > 1$ .

**Proof.** Fix any player  $i$  of type  $t_i$  and pick any message  $m_i = (m_i^1, m_i^2, m_i^3, m_i^4)$  such that  $m_i^2 > 1$ . Suppose that  $t_i$  has belief  $\lambda_i \in \Delta(\Theta \times T_{-i} \times M_{-i})$ . Let us consider  $\pi_i :=$

$\text{marg}_{\Theta \times M_{-i}} \lambda_i$ . We can partition the messages of the other agents as follows. For each  $t_{-i}$ :

$$M_{-i}^*(t_{-i}) = \{m_{-i} \mid m_j^2 = 1 \text{ for all } j \neq i \text{ and } m_{-i}^1 = t_{-i}\},$$

and

$$\hat{M}_{-i} = \{m_{-i} \mid m_j^2 > 1 \text{ for some } j \neq i\}.$$

Clearly, for  $m_{-i} \in M_{-i}^*(t_{-i})$  for some  $t_{-i}$ ,  $(m_i, m_{-i})$  falls into Rule 2 of the mechanism while for  $m_{-i} \in \hat{M}_{-i}$ ,  $(m_i, m_{-i})$  falls into Rule 3. Now, if for some  $t'_{-i}$ , we have

$$\sum_{m_{-i} \in M_{-i}^*(t'_{-i})} \pi_i(\theta, m_{-i}) > 0, \text{ set } \psi_i \in \Delta(\Theta \times \bar{T}_{-i}) \text{ such that for all } (\theta, t'_{-i}), \psi_i(\theta, t'_{-i}) := \alpha \sum_{m_{-i} \in M_{-i}^*(t'_{-i})} \pi_i(\theta, m_{-i}) \text{ where the constant } \alpha \text{ is chosen so that } \sum_{\theta, t'_{-i}} \psi_i(\theta, t'_{-i}) = 1. \text{ By}$$

Assumption 1, there exists  $y(\cdot) \in Y_i$  such that

$$\sum_{\theta, t'_{-i}} \psi_i(\theta, t'_{-i}) u_i(y(t'_{-i}), \theta) > \sum_{\theta, t'_{-i}} \psi_i(\theta, t'_{-i}) u_i(\bar{y}_i(t'_{-i}), \theta).$$

Hence, if  $m_i$  is a best-response to  $\pi_i$  (and  $\sum_{m_{-i} \in M_{-i}^*(t'_{-i})} \pi_i(\theta, m_{-i}) > 0$  for some  $t'_{-i}$ ), by construction of the mechanism, we must have

$$\begin{aligned} & \sum_{t'_{-i}} \sum_{\theta} \sum_{m_{-i} \in M_{-i}^*(t'_{-i})} \pi_i(\theta, m_{-i}) u_i(m_i^3(t'_{-i}), \theta) \\ & > \sum_{t'_{-i}} \sum_{\theta} \sum_{m_{-i} \in M_{-i}^*(t'_{-i})} \pi_i(\theta, m_{-i}) u_i(\bar{y}_i(t'_{-i}), \theta). \end{aligned} \quad (8)$$

Now, if  $\sum_{m_{-i} \in \hat{M}_{-i}} \pi_i(\theta, m_{-i}) > 0$ , set  $\phi_i(\theta) := \alpha \sum_{m_{-i} \in \hat{M}_{-i}} \pi_i(\theta, m_{-i})$  where here again, the constant  $\alpha$  is chosen so that  $\sum_{\theta} \phi_i(\theta) = 1$ . By Remark 1, there exists  $y \in \Delta(A)$  such that

$$\sum_{\theta} \phi_i(\theta) u_i(y, \theta) > \sum_{\theta} \phi_i(\theta) u_i(\hat{y}_i, \theta).$$

Hence, if  $m_i$  is a best-response to  $\pi_i$  (and  $\sum_{m_{-i} \in \hat{M}_{-i}} \pi_i(\theta, m_{-i}) > 0$ ), by construction of the mechanism, we must have

$$\sum_{\theta} \sum_{m_{-i} \in \hat{M}_{-i}} \pi_i(\theta, m_{-i}) u_i(m_i^4, \theta) > \sum_{\theta} \sum_{m_{-i} \in \hat{M}_{-i}} \pi_i(\theta, m_{-i}) u_i(\hat{y}_i, \theta). \quad (9)$$

Now, Equations (8) and (9) together show that if  $m_i = (m_i^1, m_i^2, m_i^3, m_i^4)$  were a best response, then since with probability one, either rule 2 or 3 is sparked (i.e., either  $\sum_{m_{-i} \in M_{-i}^*(t'_{-i})} \pi_i(\theta, m_{-i}) >$

0 for some  $t'_{-i}$  or  $\sum_{m_{-i} \in M_{-i}^*(t'_{-i})} \pi_i(\theta, m_{-i}) > 0$ ),  $(m_i^1, m_i^2 + 1, m_i^3, m_i^4)$  would be an even better response, a contradiction. ■

To complete the proof, it is enough to prove the following claim.

**Claim 5** Let  $\beta_i(t_i) := \{t'_i : (t'_i, 1, m_i^3, m_i^4) \in R_i(t_i \mid \mathcal{M}, \bar{T}) \text{ for some } (m_i^3, m_i^4)\}$ . The deception  $\beta := \times_i \beta_i$  is acceptable.

**Proof.** By contradiction. Assume that  $\beta$  is an unacceptable deception. By interim rationalizable monotonicity, we know that there exists  $i, t_i, t'_i \in \beta_i(t_i)$  such that for every  $\psi_i \in \Delta(\Theta \times \bar{T}_{-i} \times \bar{T}_{-i})$  satisfying

$$[1] \quad \psi_i(\theta, t_{-i}, t'_{-i}) > 0 \Rightarrow t'_j \in \beta_j(t_j) \text{ for each } j \neq i,$$

$$[2] \quad \sum_{t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) = \bar{\kappa}(t_i)[\theta, t_{-i}] \text{ for each } (\theta, t_{-i}),$$

there exists  $y^* \in Y_i$  such that

$$\sum_{\theta, t_{-i}, t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) u_i(y^*(t'_{-i}), \theta) > \sum_{\theta, t_{-i}, t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) u_i(f(t'_i, t'_{-i}), \theta).$$

Observe first that by construction of  $\beta_i$ , there exists  $m_i \in R_i(t_i \mid \mathcal{M}, \bar{T})$  such that  $m_i = (t'_i, 1, m_i^3, m_i^4)$  for some  $(m_i^3, m_i^4)$ . Now pick  $\pi_i \in \Delta(\Theta \times \bar{T}_{-i} \times M_{-i})$  such that [i]  $\pi_i(\theta, t_{-i}, m_{-i}) > 0 \Rightarrow m_j \in R_j(t_j \mid \mathcal{M}, \bar{T})$  for each  $j \neq i$  and [ii]  $\sum_{m_{-i}} \pi_i(\theta, t_{-i}, m_{-i}) = \bar{\kappa}(t_i)[\theta, t_{-i}]$  for each  $(\theta, t_{-i})$  and [iii]  $m_i \in BR_i(\text{marg}_{\Theta \times M_{-i}} \pi_i \mid \mathcal{M})$ . In addition, for all  $\theta, t_{-i}$  and  $t'_{-i}$ , set

$$\psi_i(\theta, t_{-i}, t'_{-i}) := \sum_{m_{-i} \text{ s.t. } m_{-i}^1 = t'_{-i}} \pi_i(\theta, t_{-i}, m_{-i})$$

and observe that  $\psi_i$  satisfies [1] and [2] above. Hence, there exists  $y^* \in Y_i$  such that

$$\sum_{\theta, t_{-i}, t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) u_i(y^*(t'_{-i}), \theta) > \sum_{\theta, t_{-i}, t'_{-i}} \psi_i(\theta, t_{-i}, t'_{-i}) u_i(f(t'_i, t'_{-i}), \theta)$$

which yields

$$\sum_{\theta, m_{-i}} \text{marg}_{\Theta \times M_{-i}} \pi_i(\theta, m_{-i}) u_i(y^*(m_{-i}^1), \theta) > \sum_{\theta, m_{-i}} \text{marg}_{\Theta \times M_{-i}} \pi_i(\theta, m_{-i}) u_i(f(t'_i, m_{-i}^1), \theta).$$

Thus, by construction of the mechanism, against  $\pi_i$ , message  $(t'_i, \ell, \tilde{m}_i^3, m_i^4)$  where  $\tilde{m}_i^3(\cdot) = y^*(\cdot)$  is strictly better than  $m_i = (t'_i, 1, m_i^3, m_i^4)$  for  $\ell$  large enough, which contradicts  $m_i \in BR_i(\text{marg}_{\Theta \times M_{-i}} \pi_i \mid \mathcal{M})$ . ■

## D Supplementary Material: Proof of Theorem 4

**Proof of the "If part" of Theorem 4.** Assume that  $f : \bar{T} \rightarrow A$  is rationalizable-implementable by a finite mechanism  $\mathcal{M} = (M, g)$ , i.e., that for all  $\bar{t} \in \bar{T}$ ,  $m \in R(\bar{t} \mid \mathcal{M}, \bar{T}) \implies g(m) = f(\bar{t})$ .

We first recall the following well-known lemma.

**Lemma 4 (Dekel, Fudenberg and Morris (2006))** *Fix any model  $\mathcal{T} = (T, \kappa)$  such that  $\bar{T} \subset \mathcal{T}$  and any finite mechanism  $\mathcal{M}$ . (1) For any  $\bar{t} \in \bar{T}$  and any sequence  $\{t[n]\}_{n=0}^{\infty}$  in  $T$ , if  $t[n] \rightarrow_P \bar{t}$  then, for  $n$  large enough, we have:  $R(t[n] \mid \mathcal{M}, \mathcal{T}) \subset R(\bar{t} \mid \mathcal{M}, \mathcal{T})$ . (2) For any type  $t \in T$ ,  $R(t \mid \mathcal{M}, \mathcal{T})$  is non-empty.*

Now pick any model  $\mathcal{T} = (T, \kappa)$  such that  $\bar{T} \subset \mathcal{T}$ . We show that there exists an equilibrium that continuously implements  $f$  on  $\bar{T}$ . For each player  $i$  and each type  $\bar{t}_i \in \bar{T}_i$ , fix some  $m_i(\bar{t}_i) \in R_i(\bar{t}_i \mid \mathcal{M}, \bar{T})$  and restrict the space of strategies of player  $i$  by assuming that  $\sigma_i(\bar{t}_i) = m_i(\bar{t}_i)$  for each type  $\bar{t}_i \in \bar{T}_i$ . Because  $M$  is finite and  $T$  is countable, standard arguments<sup>34</sup> show that there exists a Bayes Nash equilibrium in  $U(\mathcal{M}, \mathcal{T})$ . Let us first establish that  $\sigma$  is a Bayes Nash equilibrium in  $U(\mathcal{M}, \mathcal{T})$ . It is clear by construction that for each  $i \in \mathcal{I}$  and  $t_i \notin \bar{T}_i$ ,

$$m_i \in \text{Supp}(\sigma_i(t_i)) \implies m_i \in BR_i(\pi_i(\cdot \mid t_i, \sigma_{-i}) \mid \mathcal{M}).$$

Now fix a player  $i \in \mathcal{I}$  and a type  $\bar{t}_i \in \bar{T}_i$ . Since  $\bar{T} \subset \mathcal{T}$  is a model (and hence,  $\kappa(\bar{t}_i)$  takes its support in  $\Theta \times \bar{T}_{-i}$ ), it is easily checked that, by construction of  $\sigma$ ,  $\pi_i(m_{-i} \mid \bar{t}_i, \sigma_{-i}) > 0 \implies m_{-i} \in R_{-i}(\bar{t}_{-i} \mid \mathcal{M}, \bar{T})$  for some  $\bar{t}_{-i} \in \bar{T}_{-i}$ . Hence, by a well-known argument,  $BR_i(\pi_i(\cdot \mid \bar{t}_i, \sigma_{-i}) \mid \mathcal{M}) \subset R_i(\bar{t}_i \mid \mathcal{M}, \bar{T})$ . Since  $g(R(\bar{t} \mid \mathcal{M}, \bar{T})) = \{f(\bar{t})\}$ , we have for all  $\tilde{m}_i \in R_i(\bar{t}_i \mid \mathcal{M}, \bar{T})$ :

$$\sum_{(\theta, m_{-i}) \in \Theta \times M_{-i}} \pi_i(\theta, m_{-i} \mid \bar{t}_i, \sigma_{-i}) [u_i(g(\tilde{m}_i, m_{-i}), \theta)] = \sum_{\theta, \bar{t}_{-i}} \bar{\kappa}(\bar{t}_i)[\theta, \bar{t}_{-i}] u_i(f(\bar{t}_i, \bar{t}_{-i}), \theta),$$

and so  $BR_i(\pi_i(\cdot \mid \bar{t}_i, \sigma_{-i}) \mid \mathcal{M}) = R_i(\bar{t}_i \mid \mathcal{M}, \bar{T})$ . Hence, we must have  $m_i(\bar{t}_i) = \sigma_i(\bar{t}_i) \in BR_i(\pi_i(\cdot \mid \bar{t}_i, \sigma_{-i}) \mid \mathcal{M})$ . Thus,  $\sigma$  is a Bayes Nash equilibrium in  $U(\mathcal{M}, \mathcal{T})$  and  $\sigma|_{\bar{T}}$  is a pure Nash equilibrium in  $U(\mathcal{M}, \bar{T})$ . Now, pick any sequence  $\{t[n]\}_{n=0}^{\infty}$  in  $T$ , such that  $t[n] \rightarrow_P \bar{t}$ . It is clear that for each  $n$ :  $\text{Supp}(\sigma(t[n])) \subset R(t[n] \mid \mathcal{M}, \mathcal{T})$ . In addition, for  $n$  large enough, we know by Lemma 4 that  $R(t[n] \mid \mathcal{M}, \mathcal{T}) \subset R(\bar{t} \mid \mathcal{M}, \bar{T})$ . Then, for  $n$  large enough,  $\text{Supp}(\sigma(t[n])) \subset R(\bar{t} \mid \mathcal{M}, \bar{T})$  and so,  $(g \circ \sigma)(t[n]) = f(\bar{t})$  as claimed. ■

<sup>34</sup>The existence of a Bayes Nash equilibrium can be proved using Kakutani-Fan-Glicksberg's fixed point theorem. The space of strategy profiles is compact in the product topology. Using the fact that  $u_i : A \times \Theta \rightarrow \mathbb{R}$  is bounded, all the desired properties of the best-response correspondence (in particular upper hemicontinuity) can be established.

**Proof of the "Only if part" of Theorem 4.** We show that a social choice function  $f : \bar{T} \rightarrow A$  is continuously implementable by a countable<sup>35</sup> mechanism  $\mathcal{M}$  only if it is rationalizable implementable by some mechanism  $\mathcal{M}' \subset \mathcal{M}$  (i.e.  $M'_i \subset M_i$  for each  $i$  and  $g' = g|_{\mathcal{M}'}$ ).

Since  $f$  is continuously implementable, there exists a mechanism  $\mathcal{M} = (M, g)$  such that for any model  $\mathcal{T} = (T, \kappa)$  satisfying  $\bar{T} \subset \mathcal{T}$ , there is a Bayes Nash equilibrium  $\sigma$  in the induced game  $U(\mathcal{M}, \mathcal{T})$  where for each  $\bar{t} \in \bar{T}$ , (i)  $\sigma(\bar{t})$  is pure and (ii) for any sequence  $t[n] \rightarrow_P \bar{t}$  where for each  $n : t[n] \in T$ , we have  $(g \circ \sigma)(t[n]) \rightarrow f(\bar{t})$ . We let  $C$  be the set of pure Bayesian Nash equilibria of  $U(\mathcal{M}, \bar{T})$ . Note that because  $\bar{T}$  is finite and  $M$  is countable,  $C$  is countable. For each  $\bar{\sigma} \in C$ , we build the set of message profiles  $M(\bar{\sigma})$  in the following way.

For each player  $i$  and each positive integer  $\ell$ , we define inductively  $M_i^\ell(\bar{\sigma})$ . First, we set  $M_i^0(\bar{\sigma}) = \bar{\sigma}_i(\bar{T}_i)$ . Then, for each  $\ell \geq 1$ :

$$M_i^{\ell+1}(\bar{\sigma}) = BR_i(\Delta(\Theta \times \{\tilde{\theta}^0\} \times M_{-i}^\ell(\bar{\sigma})) \mid \mathcal{M}).$$

Recall that in the model  $\bar{\mathcal{T}} = (\bar{T}, \bar{\kappa})$ ,  $\text{marg}_{\bar{\Theta}\bar{\kappa}}(\bar{t}_i)[\tilde{\theta}^0] = 1$ , for each  $i \in \mathcal{I}$  and  $\bar{t}_i \in \bar{T}_i$ . Since  $\bar{\sigma}$  is an equilibrium in  $U(\mathcal{M}, \bar{\mathcal{T}})$ ,  $M_i^0(\bar{\sigma}) = \bar{\sigma}_i(\bar{T}_i) \subset BR_i(\Delta(\Theta \times \{\tilde{\theta}^0\} \times M_{-i}^0(\bar{\sigma})) \mid \mathcal{M}) = M_i^1(\bar{\sigma})$ . Consequently, it is clear that, for each  $\ell$ ,  $M_i^\ell(\bar{\sigma}) \subset M_i^{\ell+1}(\bar{\sigma})$ . Finally, set  $M_i(\bar{\sigma}) = \lim_{\ell \rightarrow +\infty} M_i^\ell(\bar{\sigma}) = \bigcup_{\ell \in \mathbb{N}} M_i^\ell(\bar{\sigma})$ . In the sequel, for each  $\bar{\sigma} \in C$ , we will note  $\mathcal{M}(\bar{\sigma})$  the mechanism  $(M(\bar{\sigma}), g|_{M(\bar{\sigma})})$ .

A first interesting property of the family of sets  $\{M(\bar{\sigma})\}_{\bar{\sigma} \in C}$  is the following one: there is a model  $\mathcal{T}$ , satisfying  $\bar{T} \subset \mathcal{T}$ , for which any equilibrium  $\sigma$  in  $U(\mathcal{M}, \mathcal{T})$  has full range in  $M(\sigma|_{\bar{T}})$ , i.e., each message profile in  $M(\sigma|_{\bar{T}})$  is played under  $\sigma$  at some profile of types in the model  $\mathcal{T}$ . More precisely, Proposition 3 is the first step of the proof of the "Only if part" of Theorem 4.

**Proposition 3** *There exists a model  $\mathcal{T} = (T, \kappa)$  such that for any  $\bar{\sigma} \in C$  and  $m \in M(\bar{\sigma})$ , there exists  $t[\bar{\sigma}, m] \in T$  s.t.  $\sigma(t[\bar{\sigma}, m]) = m$  for any equilibrium  $\sigma$  in  $U(\mathcal{M}, \mathcal{T})$  s.t.  $\sigma|_{\bar{T}} = \bar{\sigma}$ .*

**Proof.** We build the model  $\mathcal{T} = (T, \kappa)$  as follows. For each equilibrium  $\bar{\sigma} \in C$ , player  $i$  and integer  $\ell$ , we define inductively  $t_i[\bar{\sigma}, \ell, m_i]$  for each  $m_i \in M_i^\ell(\bar{\sigma})$  and set

$$T_i = \bigcup_{\bar{\sigma} \in C} \bigcup_{\ell=1}^{\infty} \bigcup_{m_i \in M_i^\ell(\bar{\sigma})} t_i[\bar{\sigma}, \ell, m_i] \cup \bar{T}_i.$$

Note that  $T_i$  is countable. In the sequel, we fix an arbitrary  $\bar{\sigma} \in C$ . This equilibrium  $\bar{\sigma}$  is sometimes omitted in our notations.

<sup>35</sup>As already mentioned, the "only if part" of the theorem holds beyond finite mechanisms.

For each  $\ell \geq 1$  and  $m_i \in M_i^\ell(\bar{\sigma})$ , we know that there exists  $\pi_i^{\ell, m_i} \in \Delta(\Theta \times \{\tilde{\theta}^0\} \times M_{-i}^{\ell-1}(\bar{\sigma}))$  such that  $m_i \in BR_i(\pi_i^{\ell, m_i} \mid \mathcal{M})$ . Thus we can build  $\hat{\pi}_i^{\ell, m_i} \in \Delta(\Theta \times \tilde{\Theta} \times M_{-i}^{\ell-1}(\bar{\sigma}))$  such that

$$\text{marg}_{\Theta \times M_{-i}^{\ell-1}(\bar{\sigma})} \hat{\pi}_i^{\ell, m_i} = \text{marg}_{\Theta \times M_{-i}^{\ell-1}(\bar{\sigma})} \pi_i^{\ell, m_i}$$

while  $\text{marg}_{\tilde{\Theta}} \hat{\pi}_i^{\ell, m_i} = \delta_{\tilde{\theta}^{m_i}}$ . Note that  $BR_i(\hat{\pi}_i^{\ell, m_i} \mid \mathcal{M}) = \{m_i\}$ .

In the sequel, for each player  $i$  and message  $m_i \in M_i^0(\bar{\sigma})$ , we pick one type denoted  $t_i[\bar{\sigma}, 0, m_i]$  in  $\bar{T}_i$  satisfying  $\bar{\sigma}_i(t_i[\bar{\sigma}, 0, m_i]) = m_i$ . This is well-defined because, by construction,  $M_i^0(\bar{\sigma}) = \bar{\sigma}_i(\bar{T}_i)$ . Now, for each  $\ell \geq 1$  and  $m_i \in M_i^\ell(\bar{\sigma})$ , we define inductively  $t_i[\bar{\sigma}, \ell, m_i]$  by<sup>36</sup>

$$\kappa(t_i[\bar{\sigma}, \ell, m_i])[\theta, \tilde{\theta}, t_{-i}] = \begin{cases} 0 & \text{if } t_{-i} \neq t_{-i}[\bar{\sigma}, \ell - 1, m_{-i}] \text{ for each } m_{-i} \in M_{-i}^{\ell-1}(\bar{\sigma}) \\ \hat{\pi}_i^{\ell, m_i}(\theta, \tilde{\theta}, m_{-i}) & \text{if } t_{-i} = t_{-i}[\bar{\sigma}, \ell - 1, m_{-i}] \text{ for some } m_{-i} \in M_{-i}^{\ell-1}(\bar{\sigma}) \end{cases}.$$

This probability measure is well-defined since  $\hat{\pi}_i^{\ell, m_i}(\Theta \times \tilde{\Theta} \times M_{-i}^{\ell-1}(\bar{\sigma})) = 1$ .

To complete the proof, we show that for any equilibrium  $\sigma$  of  $U(\mathcal{M}, \mathcal{T})$  such that  $\sigma|_{\bar{\mathcal{T}}} = \bar{\sigma}$ , we have:

$$\sigma_i(t_i[\bar{\sigma}, \ell, m_i]) = m_i, \quad (10)$$

for each player  $i$ , integer  $\ell$  and message  $m_i \in M_i^\ell(\bar{\sigma})$ . The proof proceeds by induction on  $\ell$ .

First note that, by construction of  $t_i[\bar{\sigma}, 0, m_i]$ , we must have for any equilibrium  $\sigma$  of  $U(\mathcal{M}, \mathcal{T})$  such that  $\sigma|_{\bar{\mathcal{T}}} = \bar{\sigma}$ :

$$\sigma_i(t_i[\bar{\sigma}, 0, m_i]) = m_i,$$

for each player  $i$  and message  $m_i \in M_i^0(\bar{\sigma})$ . Now, assume that Equation (10) is satisfied at rank  $\ell - 1$  and let us prove it is also satisfied at rank  $\ell$ . Fix any  $m_i \in M_i^\ell(\bar{\sigma})$  and any equilibrium  $\sigma$  of  $U(\mathcal{M}, \mathcal{T})$  such that  $\sigma|_{\bar{\mathcal{T}}} = \bar{\sigma}$ . Note that  $\text{Supp}(\sigma_i(t_i[\bar{\sigma}, \ell, m_i])) \subset BR_i(\pi_i \mid \mathcal{M})$  where  $\pi_i \in \Delta(\Theta \times \tilde{\Theta} \times M_{-i})$  is such that

$$\pi_i(\theta, \tilde{\theta}, m_{-i}) = \sum_{t_{-i}} \kappa(t_i[\bar{\sigma}, \ell, m_i])[\theta, \tilde{\theta}, t_{-i}] \sigma_{-i}(m_{-i} \mid t_{-i}).$$

In addition, by the inductive hypothesis and the fact that  $\sigma$  is an equilibrium of  $U(\mathcal{M}, \mathcal{T})$  satisfying  $\sigma|_{\bar{\mathcal{T}}} = \bar{\sigma}$ , we have  $\sigma_{-i}(m_{-i} \mid t_{-i}[\bar{\sigma}, \ell - 1, m_{-i}]) = 1$  for any  $m_{-i} \in M_{-i}^{\ell-1}(\bar{\sigma})$ . Hence, by construction of  $\kappa(t_i[\bar{\sigma}, \ell, m_i])$ , we have

$$\begin{aligned} \pi_i(\theta, \tilde{\theta}, m_{-i}) &= \sum_{t_{-i}} \kappa(t_i[\bar{\sigma}, \ell, m_i])[\theta, \tilde{\theta}, t_{-i}] \sigma_{-i}(m_{-i} \mid t_{-i}) \\ &= \kappa(t_i[\bar{\sigma}, \ell, m_i])[\theta, \tilde{\theta}, t_{-i}[\bar{\sigma}, \ell - 1, m_{-i}]] \\ &= \hat{\pi}_i^{\ell, m_i}(\theta, \tilde{\theta}, m_{-i}). \end{aligned}$$

<sup>36</sup> Here again, we abuse notations and write  $t_{-i}[\bar{\sigma}, 0, m_{-i}]$  for  $(t_j[\bar{\sigma}, 0, m_j])_{j \neq i}$ . Similarly,  $t[\bar{\sigma}, 0, m]$  stands for  $(t_i[\bar{\sigma}, 0, m_i])_{i \in \mathcal{I}}$ . Similar abuses will be used along this proof.

We get that  $Supp(\sigma_i(t_i[\bar{\sigma}, \ell, m_i])) \subset BR_i(\pi_i | \mathcal{M}) = BR_i(\hat{\pi}_i^{\ell, m_i} | \mathcal{M}) = \{m_i\}$  as claimed.

■

We now give a first insight on the second step of the proof. First notice that, by construction, each  $M(\bar{\sigma})$  satisfies the following closure property: taking any belief  $\pi_i \in \Delta(\Theta \times \{\bar{\theta}^0\} \times M_{-i}(\bar{\sigma}))$  such that  $BR_i(\pi_i | \mathcal{M}) \neq \emptyset$ , we must have  $BR_i(\pi_i | \mathcal{M}) \subset M_i(\bar{\sigma})$  and hence,  $BR_i(\pi_i | \mathcal{M}) = BR_i(\pi_i | \mathcal{M}(\bar{\sigma}))$ .

Now pick a type  $\bar{t}_i \in \bar{T}_i$  and a message  $m_i \in R_i^1(\bar{t}_i | \mathcal{M}(\bar{\sigma}), \bar{T})$ , it is possible to add a type  $t_i^{m_i}$  to the model  $\mathcal{T}$  defined in Proposition 3 satisfying the following two properties.<sup>37</sup> First,  $h_i^1(t_i^{m_i})$  is arbitrarily close to  $h_i^1(\bar{t}_i)$ ; second, for any equilibrium  $\sigma$  with  $\sigma_{|\bar{T}} = \bar{\sigma}$ ,  $\sigma_i(t_i^{m_i}) = m_i$ . Indeed, by definition of  $R_i^1(\bar{t}_i | \mathcal{M}(\bar{\sigma}), \bar{T})$ , there exists a belief  $\pi_i^{m_i} \in \Delta(\Theta^* \times T_{-i} \times M_{-i}(\bar{\sigma}))$  where  $\text{marg}_{\Theta^*} \pi_i^{m_i} = \text{marg}_{\Theta^*} \bar{\kappa}(\bar{t}_i)$  and such that  $m_i \in BR_i(\text{marg}_{\Theta^* \times M_{-i}(\bar{\sigma})} \pi_i^{m_i} | \mathcal{M}(\bar{\sigma}))$ . Using our assumption on cost of messages, we can slightly perturb  $\pi_i^{m_i}$  so that  $m_i$  becomes a unique best reply. So let us assume for simplicity that  $\{m_i\} = BR_i(\text{marg}_{\Theta^* \times M_{-i}(\bar{\sigma})} \pi_i^{m_i} | \mathcal{M}(\bar{\sigma}))$ . We can define the type  $t_i^{m_i}$  assigning probability  $\text{marg}_{\Theta^* \times M_{-i}(\bar{\sigma})} \pi_i^{m_i}(\theta^*, m_{-i})$  to  $(\theta^*, t_{-i}[\bar{\sigma}, m_{-i}])$  where  $t_{-i}[\bar{\sigma}, m_{-i}]$  is defined as in Proposition 3 (i.e.,  $t_{-i}[\bar{\sigma}, m_{-i}]$  plays  $m_{-i}$  under any equilibrium  $\sigma$  in  $U(\mathcal{M}, \mathcal{T})$  such that  $\sigma_{|\bar{T}} = \bar{\sigma}$ ). Now pick any equilibrium  $\sigma$  in  $U(\mathcal{M}, \mathcal{T} \cup \{t_i^{m_i}\})$  such that  $\sigma_{|\bar{T}} = \bar{\sigma}$ . By construction,  $Supp(\sigma_i(t_i^{m_i})) \subset BR_i(\text{marg}_{\Theta^* \times M_{-i}(\bar{\sigma})} \pi_i^{m_i} | \mathcal{M})$  and so  $BR_i(\text{marg}_{\Theta^* \times M_{-i}(\bar{\sigma})} \pi_i^{m_i} | \mathcal{M}) \neq \emptyset$ . By the closure property described above,  $BR_i(\text{marg}_{\Theta^* \times M_{-i}(\bar{\sigma})} \pi_i^{m_i} | \mathcal{M}) = BR_i(\text{marg}_{\Theta^* \times M_{-i}(\bar{\sigma})} \pi_i^{m_i} | \mathcal{M}(\bar{\sigma}))$  and so we get that type  $t_i^{m_i}$  plays  $m_i$  under the equilibrium  $\sigma$  and satisfies the desired property. Using a similar reasoning, we show inductively the following "contagion" result.

**Proposition 4** *There exists a model  $\hat{\mathcal{T}} = (\hat{T}, \hat{\kappa})$  such that for each equilibrium  $\bar{\sigma} \in C$  and each player  $i$  the following holds. For all  $\bar{t}_i \in \bar{T}_i$  and  $m_i \in R_i(\bar{t}_i | \mathcal{M}(\bar{\sigma}), \bar{T})$ , there exists a sequence of types  $\{\hat{t}_i[n]\}_{n=0}^\infty$  in  $\hat{T}_i$  such that (1)  $\hat{t}_i[n] \rightarrow_P \bar{t}_i$  and (2)  $\sigma_i(\hat{t}_i[n]) = m_i$  for each integer  $n$  and equilibrium  $\sigma$  of  $U(\mathcal{M}, \hat{\mathcal{T}})$  satisfying  $\sigma_{|\bar{T}} = \bar{\sigma}$ .*

**Proof.** We again define the set  $\mathcal{E}$  by:

$$\mathcal{E} := \cup_{q \in \mathbb{N}^*} \left\{ \frac{1}{q} \right\} \cup \{0\}.$$

<sup>37</sup>In this section, for any mechanism  $\mathcal{M}$ , we use the standard notation where  $R_i^\ell(\bar{t}_i | \mathcal{M}, \bar{T})$  stands for the  $\ell$ th round of elimination at type  $\bar{t}_i$  of messages that are not best responses (see for instance, Dekel, Fudenberg and Morris (2007)). Recall that for any  $\ell$ , and  $\bar{t}_i$ , we have  $R_i(\bar{t}_i | \mathcal{M}, \bar{T}) \subset R_i^\ell(\bar{t}_i | \mathcal{M}, \bar{T})$  (for additional details on the relationship between  $R_i(\bar{t}_i | \mathcal{M}, \bar{T})$  and  $R_i^\ell(\bar{t}_i | \mathcal{M}, \bar{T})$  when the set of messages is countably infinite, see Lipman (1994)).

We build the model  $\hat{\mathcal{T}} = (\hat{T}, \hat{\kappa})$  as follows. For each  $\varepsilon \in \mathcal{E}$ ,  $\ell \in \mathbb{N}^*$ ,  $\bar{\sigma} \in C$ ,  $\bar{t}_i \in \bar{T}_i$  and  $m_i \in R_i^\ell(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T})$ , we build inductively  $\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]$  and set

$$\hat{T}_i = \bigcup_{\varepsilon \in \mathcal{E}} \bigcup_{\ell=1}^{\infty} \bigcup_{\bar{\sigma} \in C} \bigcup_{\bar{t}_i \in \bar{T}_i} \bigcup_{m_i \in R_i^\ell(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T})} \hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i] \cup T_i,$$

where  $T_i$  is as defined in Proposition 3. Note that  $\hat{T}_i$  is countable. In the sequel, we fix an arbitrary  $\bar{\sigma} \in C$ . This equilibrium  $\bar{\sigma}$  is sometimes omitted in our notations.

We know that for each integer  $\ell$ , player  $i$  of type  $\bar{t}_i \in \bar{T}_i$  and message  $m_i \in R_i^\ell(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T})$ , there exists  $\pi_{\bar{t}_i}^{\ell, m_i} \in \Delta(\Theta \times \bar{T}_{-i} \times M_{-i}(\bar{\sigma}))$  such that

$$\begin{aligned} \text{marg}_{\Theta \times \bar{T}_{-i}} \pi_{\bar{t}_i}^{\ell, m_i} &= \bar{\kappa}(\bar{t}_i); \\ \text{marg}_{\bar{T}_{-i} \times M_{-i}(\bar{\sigma})} \pi_{\bar{t}_i}^{\ell, m_i}(\bar{t}_{-i}, m_{-i}) > 0 &\Rightarrow m_{-i} \in R_{-i}^{\ell-1}(\bar{t}_{-i} \mid \mathcal{M}(\bar{\sigma}), \bar{T}) \end{aligned}$$

and

$$m_i \in BR_i(\text{marg}_{\Theta^* \times M_{-i}(\bar{\sigma})} \pi_{\bar{t}_i}^{\ell, m_i} \mid \mathcal{M}(\bar{\sigma})).$$

For ease of exposition, we sometimes consider  $\pi_{\bar{t}_i}^{\ell, m_i}$  as a measure over  $\Theta \times \bar{T}_{-i} \times M_{-i}(\bar{\sigma})$  and sometimes as a measure over  $\Theta^* \times \bar{T}_{-i} \times M_{-i}(\bar{\sigma})$  assigning probability one to  $\{\bar{\theta}^0\}$ . Similar abuses will be used throughout the proof.

First, we let  $\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]$  be such that  $\hat{\kappa}(\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i])$  satisfies the following two conditions:

$$\text{marg}_{\bar{\Theta}} \hat{\kappa}(\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]) = \varepsilon \delta_{\bar{\theta}^{m_i}} + (1 - \varepsilon) \delta_{\bar{\theta}^0}. \quad (11)$$

And,

$$\text{marg}_{\Theta \times \hat{T}_{-i}} \hat{\kappa}(\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]) = \pi_{\bar{t}_i}^{1, m_i} \circ \left( \tau_{-i}^{\varepsilon, 1} \right)^{-1} \quad (12)$$

where  $\left( \tau_{-i}^{\varepsilon, 1} \right)^{-1}$  stands for the preimage of the function  $\tau_{-i}^{\varepsilon, 1} : \Theta \times \bar{T}_{-i} \times M_{-i} \rightarrow \Theta \times \hat{T}_{-i}$ , defined by  $\tau_{-i}^{\varepsilon, 1}(\theta, \bar{t}_{-i}, m_{-i}) = (\theta, t_{-i}[\bar{\sigma}, m_{-i}])$  and  $t_{-i}[\bar{\sigma}, m_{-i}] \in T_{-i}$  is the type profile defined in Proposition 3. Recall that  $\sigma_{-i}(t_{-i}[\bar{\sigma}, m_{-i}]) = m_{-i}$  for any equilibrium  $\sigma$  in  $U(\mathcal{M}, \mathcal{T})$  s.t.  $\sigma|_{\bar{T}} = \bar{\sigma}$ . Now for each  $\ell \geq 2$ , define  $\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]$  inductively by

$$\text{marg}_{\bar{\Theta}} \hat{\kappa}(\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]) = \varepsilon \delta_{\bar{\theta}^{m_i}} + (1 - \varepsilon) \delta_{\bar{\theta}^0},$$

and,

$$\text{marg}_{\Theta \times \hat{T}_{-i}} \hat{\kappa}(\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]) = \pi_{\bar{t}_i}^{\ell, m_i} \circ \left( \tau_{-i}^{\varepsilon, \ell} \right)^{-1}$$

where  $\left( \tau_{-i}^{\varepsilon, \ell} \right)^{-1}$  stands for the preimage of the function  $\tau_{-i}^{\varepsilon, \ell} : \Theta \times \bar{T}_{-i} \times M_{-i} \rightarrow \Theta \times \hat{T}_{-i}$ , defined by  $\tau_{-i}^{\varepsilon, \ell}(\theta, \bar{t}_{-i}, m_{-i}) = (\theta, \hat{t}_{-i}[\varepsilon, \ell - 1, \bar{\sigma}, \bar{t}_{-i}, m_{-i}])$ .

**Claim 6** For each  $\bar{t}_i \in \bar{T}_i$  and  $m_i \in R_i(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T}) : \hat{t}_i[\hat{\varepsilon}(\ell), \ell, \bar{\sigma}, \bar{t}_i, m_i] \rightarrow_P \bar{t}_i$  as  $\ell \rightarrow \infty$  for some mapping  $\hat{\varepsilon}$  taking values in  $\mathcal{E} \setminus \{0\}$ .

**Proof.** In the sequel, we will note  $\bar{h}$  the (continuous) mapping that projects  $\bar{T}$  into  $T^*$  and, in a similar way,  $\hat{h}$  the (continuous) mapping from  $\hat{T}$  to  $T^*$ .

For any  $\bar{t}_i \in \bar{T}_i$ , since<sup>38</sup> for all  $\ell \geq 1$  and all  $m_i \in R_i^\ell(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T}) : \hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i] \rightarrow \hat{t}_i[0, \ell, \bar{\sigma}, \bar{t}_i, m_i]$  as  $\varepsilon \rightarrow 0$ , by Lemma 2, for all  $\ell \geq 1$ , for all  $\ell' \geq 1$  and all  $m_i \in R_i^{\ell'}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T}) : \hat{h}_i^\ell(\hat{t}_i[\varepsilon, \ell', \bar{\sigma}, \bar{t}_i, m_i]) \rightarrow \hat{h}_i^\ell(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i])$  as  $\varepsilon \rightarrow 0$ .

Let us now show that for all  $\ell \geq 1$  and  $\ell' \geq \ell : \hat{h}_i^\ell(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) = \bar{h}_i^\ell(\bar{t}_i)$  for all  $\bar{t}_i \in \bar{T}_i$  and  $m_i \in R_i^{\ell'}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T})$ . First notice that the first order beliefs are equal, i.e., for all  $\ell' \geq 1$ ,  $\bar{t}_i \in \bar{T}_i$  and  $m_i \in R_i^{\ell'}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T}) :$

$$\begin{aligned} \hat{h}_i^1(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) &= \text{marg}_\Theta \hat{\kappa}(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) \\ &= \text{marg}_\Theta \pi_{\bar{t}_i}^{\ell', m_i} \circ (\tau_{-i}^{0, \ell'})^{-1} \\ &= \text{marg}_\Theta \pi_{\bar{t}_i}^{\ell', m_i} = \text{marg}_\Theta \bar{\kappa}(\bar{t}_i) = \bar{h}_i^1(\bar{t}_i) \end{aligned}$$

where the third and the fourth equalities are by definition of  $\tau_{-i}^{0, \ell'}$  and  $\pi_{\bar{t}_i}^{\ell', m_i}$  respectively. Now fix some  $\ell \geq 2$  and let  $L$  be the set of all belief profiles of players other than  $i$  at order  $\ell-1$ . Toward an induction, assume that for all  $\ell' \geq \ell-1 : \hat{h}_j^{\ell'-1}(\hat{t}_j[0, \ell', \bar{\sigma}, \bar{t}_j, m_j]) = \bar{h}_j^{\ell'-1}(\bar{t}_j)$  for each  $j$ ,  $\bar{t}_j \in \bar{T}_j$  and  $m_j \in R_j^{\ell'}(\bar{t}_j \mid \mathcal{M}(\bar{\sigma}), \bar{T})$ . Then for all  $\ell' \geq \ell : \text{proj}_{\Theta \times L} \circ (id_\Theta \times \hat{h}_{-i}) \circ \tau_{-i}^{0, \ell'} = \overline{\text{proj}}_{\Theta \times L} \circ (id_\Theta \times \bar{h}_{-i} \times id_{M_{-i}(\bar{\sigma})})$  where  $id_\Theta$  (resp.  $id_{M_{-i}(\bar{\sigma})}$ ) is the identity mapping from  $\Theta$  to  $\Theta$  (resp. from  $M_{-i}(\bar{\sigma})$  to  $M_{-i}(\bar{\sigma})$ ) while  $\text{proj}_{\Theta \times L}$  (resp.  $\overline{\text{proj}}_{\Theta \times L}$ ) is the projection mapping from  $\Theta \times T^*$  to  $\Theta \times L$  (resp. from  $\Theta \times T^* \times M_{-i}(\bar{\sigma})$  to  $\Theta \times L$ ); hence for all  $\ell' \geq \ell$ ,  $\bar{t}_i \in \bar{T}_i$  and  $m_i \in R_i^{\ell'}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T}) :$

$$\begin{aligned} \text{marg}_{\Theta \times L} \hat{\kappa}(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) \circ (id_\Theta \times \hat{h}_{-i})^{-1} &= \text{marg}_{\Theta \times L} \pi_{\bar{t}_i}^{\ell', m_i} \circ (\tau_{-i}^{0, \ell'})^{-1} \circ (id_\Theta \times \hat{h}_{-i})^{-1} \\ &= \pi_{\bar{t}_i}^{\ell', m_i} \circ (\tau_{-i}^{0, \ell'})^{-1} \circ (id_\Theta \times \hat{h}_{-i})^{-1} \circ (\text{proj}_{\Theta \times L})^{-1} \\ &= \pi_{\bar{t}_i}^{\ell', m_i} \circ (id_\Theta \times \bar{h}_{-i} \times id_{M_{-i}(\bar{\sigma})})^{-1} \circ (\overline{\text{proj}}_{\Theta \times L})^{-1} \\ &= \text{marg}_{\Theta \times L} \pi_{\bar{t}_i}^{\ell', m_i} \circ (id_\Theta \times \bar{h}_{-i} \times id_{M_{-i}(\bar{\sigma})})^{-1} \\ &= \text{marg}_{\Theta \times L} \bar{\kappa}(\bar{t}_i) \circ (id_\Theta \times \bar{h}_{-i})^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{h}_i^\ell(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) &= \delta_{\hat{h}_i^{\ell-1}(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i])} \times \text{marg}_{\Theta \times L} \hat{\kappa}(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) \circ (id_\Theta \times \hat{h}_{-i})^{-1} \\ &= \delta_{\bar{h}_i^{\ell-1}(\bar{t}_i)} \times \text{marg}_{\Theta \times L} \bar{\kappa}(\bar{t}_i) \circ (id_\Theta \times \bar{h}_{-i})^{-1} = \bar{h}_i^\ell(\bar{t}_i) \end{aligned}$$

<sup>38</sup>A type in  $\hat{T}_i$  is either in  $T_i$  – which is endowed with the discrete topology, say  $\tau_{T_i}$  – or in  $\hat{T}_i \setminus T_i$ . Any point in  $\hat{T}_i \setminus T_i$  is identified with an element of the set  $\mathcal{E} \times \mathbb{N} \times C \times \bar{T}_i \times M_i$  where  $\mathbb{N}, C, \bar{T}_i, M_i$  are all endowed with the discrete topology while  $\mathcal{E}$  is endowed with the usual topology on  $\mathbb{R}$  induced on  $\mathcal{E}$ . Finally,  $\mathcal{E} \times \mathbb{N} \times C \times \bar{T}_i \times M_i$  is endowed with the product topology; call this topology  $\tau_{\hat{T}_i \setminus T_i}$ . The topology over  $\hat{T}_i$  is the coarsest topology that contains  $\tau_{T_i} \cup \tau_{\hat{T}_i \setminus T_i}$ . It can easily be checked that under such a topology,  $\hat{T}$  satisfies the conditions of Lemma 2.

showing that  $\hat{h}_i^\ell(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) = \bar{h}_i^\ell(\bar{t}_i)$ . Thus, we have proved that for all  $\ell \geq 1$ , all  $\ell' \geq \ell : \hat{h}_i^\ell(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) = \bar{h}_i^\ell(\bar{t}_i)$  for any  $\bar{t}_i \in \bar{T}_i$  and  $m_i \in R_i^{\ell'}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T})$ , i.e.,  $\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i] \rightarrow_P \bar{t}_i$  as  $\ell' \rightarrow \infty$  for any  $\bar{t}_i \in \bar{T}_i$  and  $m_i \in R_i(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T})$ . In addition, we know that for all  $\ell' \geq 1$  and all  $m_i \in R_i(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T}) : \hat{t}_i[\varepsilon, \ell', \bar{\sigma}, \bar{t}_i, m_i] \rightarrow_P \hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]$  as  $\varepsilon \rightarrow 0$ . Since  $T^*$  is a metrizable space,  $\hat{t}_i[\hat{\varepsilon}(\ell'), \ell', \bar{\sigma}, \bar{t}_i, m_i] \rightarrow_P \bar{t}_i$  as  $\ell' \rightarrow \infty$  for some function  $\hat{\varepsilon} : \mathbb{N}^* \rightarrow \mathcal{E} \setminus \{0\}$  satisfying  $\lim_{\ell' \rightarrow \infty} \hat{\varepsilon}(\ell') = 0$ . ■

**Claim 7** For each  $\varepsilon \in \mathcal{E} \setminus \{0\}$ ,  $\ell, \bar{t}_i \in \bar{T}_i$  and  $m_i \in R_i(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T})$ , we have:  $\sigma_i(\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]) = m_i$  for any equilibrium  $\sigma$  of  $U(\mathcal{M}, \hat{T})$  satisfying  $\sigma|_{\bar{T}} = \bar{\sigma}$ .

**Proof.** Fix a type  $\bar{t}_i \in \bar{T}_i$  and an equilibrium  $\sigma$  of  $U(\mathcal{M}, \hat{T})$  satisfying  $\sigma|_{\bar{T}} = \bar{\sigma}$ . We will show by induction on  $\ell$  that for all  $\varepsilon \in \mathcal{E} \setminus \{0\}$  and  $\ell \geq 1$ :  $\sigma_i(\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]) = m_i$  for all message  $m_i \in R_i^\ell(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T})$ .

Recall that, by construction, for all  $m_i \in M_i(\bar{\sigma})$ ,  $t_i[\bar{\sigma}, m_i] \in T_i$  is the type in Proposition 3 such that  $\sigma_i(t_i[\bar{\sigma}, m_i]) = m_i$ . First, fix  $\varepsilon \in \mathcal{E} \setminus \{0\}$  and  $m_i \in R_i^1(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T})$  and let us prove that  $\sigma_i(\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]) = m_i$ . For each  $\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]$ , define the belief

$$\pi_i^{\varepsilon, 1} = \hat{\kappa}(\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]) \circ \gamma^{-1} \in \Delta(\Theta^* \times \hat{T}_{-i} \times M_{-i}),$$

where  $\gamma : (\theta^*, t_{-i}[\bar{\sigma}, m_{-i}]) \mapsto (\theta^*, t_{-i}[\bar{\sigma}, m_{-i}], m_{-i})$ . Note that by construction,  $\pi_i^{\varepsilon, 1}$  is the belief of type  $\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]$  on  $\Theta^* \times \hat{T}_{-i} \times M_{-i}$  when he believes that  $m_{-i}$  is played at each  $(\theta^*, t_{-i}[\bar{\sigma}, m_{-i}])$ . Hence, for each  $\varepsilon \geq 0$ ,  $\pi_i^{\varepsilon, 1}$  corresponds to beliefs of type  $\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]$  when the equilibrium  $\sigma$  is played. Now, by Equations (11) and (12), the belief  $\pi_i^{0, 1}$  of type  $\hat{t}_i[0, 1, \bar{\sigma}, \bar{t}_i, m_i]$  satisfies

$$\text{marg}_{\Theta^* \times M_{-i}} \pi_i^{0, 1} = \text{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1, m_i} \circ \left( \tau_{-i}^{0, 1} \right)^{-1} \circ (\gamma_\Theta)^{-1} = \text{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1, m_i},$$

where  $\gamma_\Theta : (\theta, t_{-i}[\bar{\sigma}, m_{-i}]) \mapsto (\theta, \tilde{\theta}^0, t_{-i}[\bar{\sigma}, m_{-i}], m_{-i})$ . Since  $\text{Supp}(\sigma_i(\hat{t}_i[0, 1, \bar{\sigma}, \bar{t}_i, m_i])) \subset BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{0, 1} \mid \mathcal{M})$ , we have:  $BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1, m_i} \mid \mathcal{M}) \neq \emptyset$ . In addition, since  $\text{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1, m_i}(\Theta \times \{\tilde{\theta}^0\} \times M_{-i}(\bar{\sigma})) = 1$ , by construction of  $M_i(\bar{\sigma})$  we have  $BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1, m_i} \mid \mathcal{M}) \subset M_i(\bar{\sigma})$ . Thus,

$$BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1, m_i} \mid \mathcal{M}(\bar{\sigma})) = BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1, m_i} \mid \mathcal{M}).$$

Recall that, by construction of  $\pi_{\bar{t}_i}^{1, m_i}$ ,  $m_i \in BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1, m_i} \mid \mathcal{M}(\bar{\sigma}))$ . Consequently,

$$m_i \in BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi_i^{0, 1} \mid \mathcal{M}).$$

In addition, we have

$$\text{marg}_{\Theta \times M_{-i}} \pi_i^{\varepsilon, 1} = \text{marg}_{\Theta \times M_{-i}} \pi_i^{0, 1}.$$

Hence, for  $\varepsilon \in \mathcal{E} \setminus \{0\}$ , by construction of  $\pi_i^{\varepsilon,1}$ ,  $\{m_i\} = BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi_i^{\varepsilon,1} \mid \mathcal{M})$  and  $\sigma_i(\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]) = m_i$ .

Now, for each  $\ell \geq 2$ , proceed by induction and assume that  $\sigma_{-i}(\hat{t}_{-i}[\varepsilon, \ell-1, \bar{\sigma}, \bar{t}_{-i}, m_{-i}]) = m_{-i}$  for any  $\bar{t}_{-i} \in \bar{T}_{-i}$ ,  $m_{-i} \in R_{-i}^{\ell-1}(\bar{t}_{-i} \mid \mathcal{M}(\bar{\sigma}), \bar{T})$  and  $\varepsilon \in \mathcal{E} \setminus \{0\}$ . Fix  $\varepsilon \in \mathcal{E} \setminus \{0\}$  and  $m_i \in R_i^\ell(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{T})$ . For each  $\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]$ , define the belief

$$\pi_i^{\varepsilon, \ell} = \hat{\kappa}(\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]) \circ \gamma_\ell^{-1} \in \Delta(\Theta^* \times \hat{T}_{-i} \times M_{-i})$$

where  $\gamma_\ell : (\theta^*, \hat{t}_{-i}[\varepsilon, \ell-1, \bar{\sigma}, \bar{t}_{-i}, m_{-i}]) \mapsto (\theta^*, \hat{t}_{-i}[\varepsilon, \ell-1, \bar{\sigma}, \bar{t}_{-i}, m_{-i}], m_{-i})$ .

Note that, by construction,  $\pi_i^{\varepsilon, \ell}$  is the belief of type  $\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]$  on  $\Theta^* \times \hat{T}_{-i} \times M_{-i}$  when he believes that  $m_{-i}$  is played at each  $(\theta^*, \hat{t}_{-i}[\varepsilon, \ell-1, \bar{\sigma}, \bar{t}_{-i}, m_{-i}])$ . Hence, by the induction hypothesis, for each  $\varepsilon \geq 0$ ,  $\pi_i^{\varepsilon, \ell}$  corresponds to beliefs of type  $\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]$  when the equilibrium  $\sigma$  is played. The end of the proof mimics the case  $\ell = 1$ . ■

This completes the proof of Proposition 4. ■

**Completion of the proof of the "Only if part" of Theorem 4.** Pick  $\hat{T} = (\hat{T}, \hat{\kappa})$  as defined in Proposition 4. By definition of continuous implementation, there exists an equilibrium  $\sigma$  in  $U(\mathcal{M}, \hat{T})$  that continuously implements  $f$  and point (i) in this definition ensures that  $\sigma_{|\hat{T}}$  is a pure equilibrium. Now pick any  $\bar{t} \in \bar{T}$  and  $m \in R(\bar{t} \mid \mathcal{M}(\sigma_{|\hat{T}}), \bar{T})$ , we show that  $g_{|\mathcal{M}(\sigma_{|\hat{T}})}(m) = f(\bar{t})$  proving that the mechanism  $\mathcal{M}(\sigma_{|\hat{T}})$  implements  $f$  in rationalizable messages. Applying Proposition 4, we know that there exists a sequence of types  $\{\hat{t}[n]\}_{n=0}^\infty$  in  $\hat{T}$  such that (1)  $\hat{t}[n] \rightarrow_P \bar{t}$  and (2)  $\sigma(\hat{t}[n]) = m$  for all  $n$ . By (1) and the fact that  $\sigma$  continuously implements  $f$ , we have  $(g \circ \sigma)(\hat{t}[n]) \rightarrow f(\bar{t})$  while by (2) we have  $(g \circ \sigma)(\hat{t}[n]) = g(m)$  for all  $n$ . Hence, we must have  $g(m) = f(\bar{t})$  and so  $g_{|\mathcal{M}(\sigma_{|\hat{T}})}(m) = f(\bar{t})$  as claimed.