0.1 THE PRINCIPLES OF GAME THEORY

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0.2 Introduction

The prehistory of game theory is relatively short, devoted to an algorithm for the resolution of extensive-form games (Zermelo) and an equilibrium notion for zero-sum normal-form games (Borel, von Neumann). The theory appeared in an already elaborated form in the pioneering work "Theory of Games and Economic Behavior" (1944), issued from the collaboration between the mathematician von Neumann and the economist Morgenstern. It saw a first period of development during the 1950s, when its main equilibrium concept - Nash equilibrium - was introduced (Nash). It then fell into neglect for a time, before enjoying a second burst of life in the 1970s with the explicit integration of time (Selten) and uncertainty (Harsanyi) into the equilibrium notions. It enjoyed a further boost in the 1990s, with the explicit internalization of players’ beliefs (Aumann) and the advent of evolutionary game theory.

The central purpose of game theory is to study the strategic relations between supposedly rational players. It thus explores the social structures within which the consequences of a player’s action depend, in a conscious way for the player, on the actions of the other players. To do so, it takes as its basis the rational model of individual decision, although current work is increasingly focused on the limited rationality of the players. It studies direct multilateral relations between players, non-mediatized by prior institutions. Game theory is generally divided into two branches, although there are bridges that connect the two. Noncooperative game theory studies the equilibrium states that can result from the autonomous behavior of players unable to define irrevocable contracts. Cooperative game theory studies the results of games governed by both individual and collective criteria of rationality, which may be imposed by an agent at some superior level.

Game theory’s natural field of application is economic theory: the economic system is seen as a huge game between producers and consumers, who transact through the intermediation of the market. It can be more specifically applied to situations outside the realm of perfectly competitive markets, i.e. situations in which the agents acquire some power over the fixing of prices (imperfect competition, auction mechanisms, wage negotiations). It can be applied equally well to relations between the state and agents, or to relations between two states. Nonetheless, it is situated at a level of generality above that of economic theory, for it considers non-
specialized - though heterogeneous - agents performing actions of one na-
ture or another within an institution-free context. It can therefore be con-
sidered a general matrix for the social sciences and be applied to social
relations as studied in political science, military strategy, sociology, or even
relations between animals in biology.

In this paper, we shall restrict ourselves to the study of noncooperative
game theory. This is the prototype of formalized social science theory which,
though making enormous use of diverse mathematical tools, transmits very
simple, even simplistic, literary messages. From a syntactic point of view, it
provides explanations of some social phenomena on the basis of a small
number of concepts and mechanisms, thus endowing these phenomena with
a duly signposted domain of validity. From a semantic point of view, it is
the subject of laboratory experimentation, although this is aimed more
at testing the consequences of existing theories than at inducing original
regularities from the results. From a pragmatic point of view, it provides
a unifying language for parties faced with decision-making, helping them
better to express their shared problems, but supplying few instructions able
to help them solve these problems.

In what follows, we shall only consider two-player games, for the sake of
simplicity (thus avoiding the question of coalitions between players). The
games will be examined not only from the point of view of the modelizer
overlooking them but also from the points of view of the players them-
selves. Each player is characterized by three "choice determinants": his
opportunities (sets of possible actions), his beliefs (representations of the
environment) and his preferences (value judgments on the effects of ac-
tions). Interaction between the players is liable to lead to an "equilibrium
state" defined - as it is in mechanics - as a situation that remains stable in
the absence of perturbations from the environment. We shall start with the
simplest forms of game and make the model more complex by the gradual
introduction of the concepts of time and uncertainty. In each section, we
shall present behavior assumptions illustrated by simple examples and then
deal with the equilibrium concepts introduced with their properties.

0.3 Static games without uncertainty

0.3.1 Behavior assumptions

We assume that it is possible to isolate a game clearly delimited from the
complex situations in which the players are immersed and particularly from
the parallel or sequential games in which they may participate. In other
words, the description of the game must be self-sufficient in the sense that
no external consideration can be introduced subsequently to account for
the game. The choice determinants for player i (internalizing the rules of
the game) are expressed in the following manner:
- a set $A_i$ of possible actions $a_i$
- perfect beliefs concerning the structure of the game, connecting the actions $a_i$ to the common consequences $r_{ij}$ by the relation $r_{ij} = F(a_i, a_j)$
- a utility function defined for the consequences of the actions: $U_i(r_{ij})$;
  however, as these consequences are commonly known as a function of the actions chosen, the utility function can be directly defined on actions by bypassing the consequences: $U_i(a_i, a_j)$.

The game is "finite" if the number of actions per player is finite. A finite two-player game is usually represented in a "strategic form" (or "normal form"), in other words a bimatrix of the game, where the rows correspond to the actions of the first player and the columns to the actions of the second player. At the intersection of a row and a column, that is to say for a pair of actions, the "outcome" of the game is indicated, constituted of the utility obtained by each player. The game matrix is assumed to be known by both players. Some classes of games (finite or infinite) are worth mentioning at this point. Zero-sum games are games in which the sum of the utilities of the players is zero (or constant) for every outcome (totally divergent interests). Twin games are games in which the utilities of the players are identical for each outcome (totally convergent interests). Symmetric games are obtained when the players’ action sets are identical and when the players, by exchanging their actions, also exchange their utilities.

### 0.3.2 Elementary games

The most famous finite game is the “prisoner’s dilemma”. This symmetric game involves two prisoners, in non-communicating cells, who are believed to have committed a crime together. Each prisoner can either confess (thus denouncing the other) or not confess, knowing that the institutional system in place encourages denouncement. If one confesses but not the other, the first is freed and the second is given the maximum sentence. If neither confess, they both receive light sentences. If they both confess, they both receive sentences that are heavy, but below the maximum. The game matrix is obtained by valuating utilities in inverse proportion to the prison sentence, giving these utilities the values 0, 1, 2 and 3:

$$
\begin{array}{c|cc}
 & \text{confess} & \text{not confess} \\
\hline
\text{confess} & (1,1) & (3,0) \\
\text{not confess} & (0,3) & (2,2) \\
\end{array}
$$

Two other finite games are classics. The "battle of the sexes" is a symmetric game in which the husband (player 1) and the wife (player 2) have a choice of two outings, either the ballet or the boxing match. Each player gives the value of 1 to the utility of going to their preferred form of entertainment (the boxing match for the husband, the ballet for the wife) and
the added value of 2 to the utility of being together. The "war game" is a zero-sum game in which two opposing armies, the attacker (player 1) and the defender (player 2), can position their troops in either place A or place B. If the troops are in the same place, the defender wins (utility 1 for the defender, 0 for the attacker); if they are in different places, the attacker wins (utility of 1 for the attacker, 0 for the defender).

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A representative infinite game is the "duopoly game", which opposes two firms in imperfect competition on a market for one type of good. Each firm i can freely fix its level of production $q_i$ and seeks to maximize its profits, which depend on the price $p$ of the good and on the marginal cost (constant) $c_i$: $\Pi_i = pq_i - c_iq_i = q_i(p - c_i)$. The price is determined by balancing the supply: $q = q_1 + q_2 = O(p)$ and the demand: $q = D(p)$. The quantity demanded is assumed to be a decreasing linear function of the price if $p$ is below a certain threshold and zero if the price is above this threshold. The inverse demand $p = D^{-1}(q)$ is such that the price is undetermined (above a certain threshold) if the quantity is zero, and is then a decreasing linear function of the quantity (until it reaches zero). In the useful zone, therefore, we have: $p = a - b(q_1 + q_2)$. The profit of firm i can thus be written: $\Pi_i = q_i(a - b(q_1 + q_2) - c_i)$. This game cannot be represented by a matrix. However, it is still possible to consider matrices with specific values of quantities produced.

0.3.3 Strategies

The choice of a player revolves around his "strategies" $s_i$. A "pure strategy" is nothing other than an action: $s_i = a_i$. A "mixed strategy" is a probability distribution on the actions: $s_i = \sigma_i$, where $\sigma_i(a_i)$ is the probability allocated to the action $a_i$. The actions to which a strictly positive probability is assigned are called the support of the mixed strategy. It is assumed, provisionally, that a player implements a mixed strategy by drawing at random an action from the corresponding probability distribution. He gains from a mixed strategy the expected utility on all outcomes, evaluated with the (independent) probability distributions of the player and his opponent. If the use of pure strategies requires only that the utilities must be ordinal (the utility of a player is defined up to an increasing function), the use of mixed strategies requires that the utilities must be cardinal (the utility of a player is defined up to an affine increasing function). It should be noted that the rule of maximization of expected utility, natural in a choice against nature assumed to be passive, has been transposed to the choice against a rational and therefore active opponent, thus "naturalizing" the other player.
The possible outcomes of a finite game can be represented by a domain in the players’ utility space (a set of points in the case of pure strategies, an often continuous domain in the case of mixed strategies). These outcomes can be compared with the help of a very simple social criterion. One outcome Pareto-dominates another if it provides both players with a greater utility (and strictly greater for at least one of the two players). An outcome is Pareto-optimal if it is not Pareto-dominated by any other outcome. There is always at least one Pareto-optimal outcome, and they are generally multiple. In the players’ utility space, the Pareto-optimal outcomes correspond to the points on the North-East frontier of possible outcomes. From the modelizer’s point of view, it appears highly desirable to achieve a Pareto-optimal outcome. But in a noncooperative game, in which the players cannot sign binding agreements, nothing guarantees that such an outcome will actually be achieved through an equilibrium.

0.3.4 Dominant strategy equilibrium

The first leading principle is that of individual dominance between the pure strategies of a player. A player’s pure strategy $s_i$ dominates weakly (strongly) another strategy $s_i'$ of the same player if it provides him with a (strictly) greater utility against any of the opponent’s strategies (against any defense): $U_i(s_i, s_j) \geq U_i(s_i', s_j), \forall s_j$. A player’s strategy is said to be weakly (strongly) dominant if it (strongly) dominates all other strategies. A strategy is said to be (strongly) dominated if it is (strongly) dominated by another strategy. The concept of individual dominance can easily be transferred from pure strategies to mixed strategies. One can observe, however, that a player’s strategy can be dominated by a mixed strategy of this player without being dominated by any pure strategy. Above all, it should be noted that a player’s dominant strategy is defined without any reference to the precise action of his opponent; the player can therefore calculate his dominant strategy without making any assumptions about the behavior of the other.

A first equilibrium concept is that of “equilibrium in weakly (strongly) dominant strategies”. The associated equilibrium state is defined as an outcome in which each player plays a dominant strategy. Of course, such an equilibrium state may not exist. If it does exist, it is generally unique when dominance is strong but may be multiple when dominance is weak. A weaker concept is that of “sophisticated equilibrium”. It is obtained by considering an iterated process of elimination of dominated strategies. In the first round, the dominated strategies of each player are eliminated; in the second round, in the remaining game, the dominated strategies are once again eliminated; the process stops when there are no more dominated strategies. This process is univocal for the elimination of strongly dominated strategies, but not for weakly dominated strategies. In this latter case, the order of treatment of the players matters and if two strategies
are equivalent for a player, the one that is eliminated is chosen arbitrarily. Consequently, a sophisticated equilibrium state is an outcome that survives this process of sequential elimination of dominated strategies. There always exists a sophisticated equilibrium, but they may be multiple.

In the prisoner’s dilemma, confessing is a strongly dominant strategy for both players, for each player has something to gain from confessing, whatever the other player chooses to do. Thus there exists a unique equilibrium state in dominant strategies, in which both players confess. This outcome is also the unique sophisticated equilibrium. However, it is not Pareto-optimal, for the players obtain greater utility if neither of them confess. This divorce between equilibrium and Pareto-optimality raises the fundamental “problem of cooperation” between the players. In the battle of the sexes or the war game, there is no dominant or dominated strategy. The four possible outcomes are thus sophisticated equilibria. In the duopoly game, neither player has a dominant strategy. But some strategies are strongly dominated, namely those that lead to a price above the threshold at which the demand reaches zero. It is therefore possible to prove, by iteration, that a unique sophisticated equilibrium exists (at least if one restricts the game to two firms).

0.3.5 Nash equilibrium

The second leading principle is that of “best reply”. A player’s pure strategy is the best reply to the opponent’s strategy if it results in the maximization of the player’s utility, with the opponent’s strategy being fixed: \( s_i = R_i(s_j) = \arg \max \, U_i(s_i, s_j) \). There is always a best reply to an opponent’s strategy, but they may be multiple: \( R_i \) is the best reply correspondence. A strategy that is never a best reply, for any of the opponent’s strategies, is said to be inferior. This concept can also be easily transferred to mixed strategies. A player’s mixed strategy is the best reply to an opponent’s mixed strategy if and only if each strategy of its support is in itself a best reply to the mixed strategy of the opponent. However, a player’s mixed strategy may be the best reply to a mixed strategy of the opponent without being the best reply to any of the opponent’s pure strategies. Above all, it should be noted that the concept of best reply assumes that each player forms a prediction of the opponent’s strategy, called a “conjecture”.

The second concept of equilibrium is that of “Nash equilibrium”. It defines an equilibrium state \((s_i*, s_j*)\) as an outcome where each player plays his best reply to the equilibrium strategy of the other: \( U_i(s_i*, s_j*) \geq U_i(s_i, s_j*), \forall_{s_i} \). In such a state, no player has any interest in deviating unilaterally, a minimal stability condition for players acting freely and in isolation. A Nash equilibrium is said to be “strict” if the preceding inequalities are strict. The equilibrium states in pure strategies may be strict or not; the equilibrium states in mixed strategies are always non-strict. An equilibrium state appears as a fixed point of a loop of best reply functions.
(correspondences), each strategy of one player being a best reply to the (expected) strategy of the other. Here again, a weaker concept exists, that of “rationalizable equilibrium”. A rationalizable equilibrium state is defined by the fact that each player plays his best reply to the expected strategy of the other player, which is itself expected to be the best reply to some strategy of the first, and so on, until the circle closes at some level. Nash equilibrium is a particular case of rationalizable equilibrium when the circle closes at the very first level. It can also be demonstrated that if only one rationalizable equilibrium exists, then it is also the sole Nash equilibrium.

However, introduced in this way, the concept of Nash equilibrium is not constructive: the process by which the players can reach an equilibrium state is not made explicit. Everything takes place as if some kind of “Nashian regulator” existed (along the lines of the “Walrasian auctioneer” of economic theory) who calculates the equilibrium and suggests it to the players (which is not sufficient to establish it). In addition, the concept of equilibrium is incomplete: several equilibrium states may exist, between which the theory cannot choose. In a finite game, pure strategy Nash equilibria may well not exist, just as they may be multiple. In an infinite game, sufficient conditions for the existence of a pure strategy Nash equilibrium have been established: the set of possible actions of a player is a compact, convex subset of a finite dimensional space on reals; the utility function of a player is continuous in relation to the actions of the players and quasi-concave in relation to the action of this player. In a finite game, there is always at least one mixed strategy equilibrium (the conditions defined above are met). A mixed strategy Nash equilibrium can be calculated by expressing that all the pure strategies of the support of a player’s mixed strategy equilibrium state give the same utility against the opponent’s mixed strategy at equilibrium.

In the prisoner’s dilemma, the only Nash equilibrium (in pure or mixed strategies) is the outcome in which both players confess. In the battle of the sexes, two asymmetrical Nash equilibria exist in pure strategies, corresponding to the outcomes in which husband and wife go to the same show (an additional symmetrical equilibrium exists in mixed strategies). What occurs here is a “problem of co-selection” for which one of the two equilibria must be chosen, knowing that the first is more favorable to one player and the second is more favorable to the other player. In the war game, there are no Nash equilibria in pure strategies, which raises a “problem of compatibility”; however, there is a Nash equilibrium in mixed strategies, with each player choosing A and B with equiprobability. In the duopoly game, with identical marginal costs $c_i$ for the two firms ($c_i = c$), there is one sole Nash equilibrium in pure strategies, the “Cournot equilibrium”, which coincides with the sophisticated equilibrium, in which the productions are: $q_1 = q_2 = (a - c)/3b$. One can also define a “collusive equilibrium” when, contrary to the founding hypothesis of non-cooperative game, the two firms can unite to form a monopoly in the face of the consumers, in which case
the productions are: \( q_1 = q_2 = (a - c)/4b \).

0.4 Dynamic games without uncertainty

0.4.1 Behavior assumptions

We will now consider that the game develops sequentially over a given horizon, and that one sole player plays in each period. The choice determinants of the agents are now more complicated:

- the opportunities of action are defined each time the player has a move and they may depend on the past history of the game, i.e. all the moves that have already been made
- the beliefs of each player are still perfect, but they cover not only the structure of the game (characteristics of opponents, horizon of the game), but also the unfolding of the game (actions taken in the past)
- the player’s preferences are only defined globally over the whole history of the game; even if they are defined for each period, they are aggregated along the path followed by the play.

The game is finite if it allows a finite number of possible histories, with a finite horizon if all histories have a finite horizon. A finite game can be expressed in “extensive form” (or “developed form”) by a “game tree” (a tree is a connected graph with no cycles):

- the non-terminal nodes correspond to the instants when it is the turn of one player to move and the edges that start from this node are the possible actions of this player
- each global history of the game corresponds to one of the possible paths in the game tree and ends in a terminal node
- the terminal nodes indicate the outcomes of the game, i.e. the overall utilities obtained by the players, through inclusion of all the partial utilities obtained along the corresponding path.

0.4.2 Elementary games

The “entry game”, also known as the “chain-store paradox”, is a finite game in which an incumbent monopolist (player 2) faces a firm that could enter into the same market (player 1). The potential entrant can decide to enter or not to enter; if he enters, the monopolist can decide to be aggressive or pacific. If the potential entrant decides not to enter, the monopolist obtains a utility of 2 corresponding to the value of the market and the potential
entrant obtains a utility of 0; if the potential entrant decides to enter and the monopolist is pacific, they share the market (utility of 1 for each player), whereas if the monopolist is aggressive they embark on a price war (utility of -1 for each player):

\[
\begin{array}{c}
E \\
1 \rightarrow \rightarrow \rightarrow 2 \\
\downarrow \downarrow \downarrow \\
(0,2) \quad (1,1) \quad (-1,-1)
\end{array}
\]

The “centipede game” is a finite game in which two players are to share a kitty, the contents of which increase by 2 units in each period. At any moment the player whose move it is can decide to continue or to stop the game. If he stops, he has a larger share of the kitty than his opponent. The “three-legged” version is illustrated below. We observe that it is always the last two outcomes that are Pareto-optimal.

\[
\begin{array}{c}
C \\
1 \rightarrow \rightarrow \rightarrow 2 \\
\downarrow \downarrow \downarrow \\
(2,0) \quad (1,3) \quad (4,2) \quad (3,5)
\end{array}
\]

The “ultimatum game” is an infinite game (but with a finite horizon) concerning the sharing of a cake of a given size. The first player makes a proposal on how to divide the cake (a fraction x for himself and a fraction 1 - x for the other player); the second player can then accept or refuse this proposal. If he accepts, the cake is shared out in accordance with the first player’s proposal; if he refuses then nobody gets any cake. The first player has an infinite number of possible actions and the second player reacts to each of them, but the game can nevertheless be illustrated by a tree with a slight abuse of notation:

\[
\begin{array}{c}
X \\
1 \rightarrow \rightarrow \rightarrow 2 \rightarrow \rightarrow \rightarrow (x,1-x) \\
\downarrow \downarrow \\
(0,0)
\end{array}
\]

0.4.3 Strategies

Here, the concept of (pure) strategy of a player finds its full significance and designates the action played instantaneously in relation to the whole past history. Thus, by defining a strategy, the player describes in advance the different actions he will undertake in each circumstance in which he could find himself. If he can communicate his strategy, the player can even
delegate his behavior to a passive entity that mechanically implements the desired action. As for the concept of mixed strategy, it can be introduced in two different forms. A “random strategy” is obtained by considering a probability distribution on pure strategies. A “behavioral strategy” is obtained by considering, at each node where it is the player’s move, a probability distribution on the actions possible on this node. These two concepts coincide if the player has a perfect memory, i.e. if he remembers his past actions perfectly. However, the second concept is better at conserving the time aspect of the game and the decentralization of choices that goes with it.

Thanks to the concept of strategy, all extensive-form finite games can be transformed into strategic-form games. In a tree, each couple of (pure) strategies of the players leads to a well defined terminal node and therefore to a precise outcome. The game matrix is thus formed by putting the strategies of the two players into lines and columns and writing the corresponding utilities of the players into each box at their intersections. In the opposite direction, one matrix could come from different trees; but in this case the strategic and extensive forms are nevertheless strategically equivalent. Above all, one cannot find a corresponding tree for every matrix; the reason is that a tree assumes that the players have perfect knowledge of past moves, whereas a matrix allows for players to play simultaneously without knowing the opponent’s move.

In the entry game, as each player only plays in a single node, each strategy coincides with an action; the strategic form is illustrated below. In the centipede game, the second player also only plays once and his strategies therefore coincide with his actions; the first player is liable to play twice and he possesses four strategies, obtained by combining the choices of stopping or continuing on the first and on the third node; note that the strategy foresees what the player would do on the third node even if he has in fact stopped the game on the first node, in other words if he were somehow to be involuntarily parachuted to the third node. In the ultimatum game, the first player’s strategy is his proposal for dividing up the cake; the second player’s strategy is to set a threshold X such that he will accept the first player’s proposal if x is below the threshold, otherwise he will reject it.

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<td>E</td>
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0.4.4 Nash equilibrium and subgame-perfect equilibrium

In the strategic form associated with the extensive form of the game, the concept of Nash equilibrium can still be defined. A Nash equilibrium obeys the principle of “conditional optimality”: on whatever node a player may be, he never has anything to gain from deviating unilaterally from his local move (both his own and his opponent’s strategies being fixed). However,
this concept has nothing to say about what happens outside the equilibrium path. In particular, this may rely on incredible threats (or promises). A threat is a sub-strategy announcing the move that one player would make if he were on a certain node, with the aim of dissuading the other player from arriving on this node; a threat is incredible if the player making it has nothing to gain from carrying it out if the node in question is actually reached. We can eliminate these incredible threats by using a stronger concept than that of Nash equilibrium, one which strengthens the property of conditional optimality for a game with finite horizon: a player has nothing to gain by deviating unilaterally from a local move on any of the nodes of the game, whether or not they are on the equilibrium path.

A state that is a Nash equilibrium in the initial game and in every sub-game (a subgame being simply a truncated game that starts in any situation where a player has to move) is called a “subgame-perfect equilibrium” (abbreviated to “perfect equilibrium”, although this latter term exists with a different meaning elsewhere). For a finite game in extensive form, perfect equilibrium is obtained by a “backward induction” procedure on the game tree: on the nodes preceding the terminal nodes, the players concerned choose their best actions; on the anti-penultimate nodes, the players concerned choose their best actions, taking into account the actions to be chosen subsequently; and so on until arriving at the initial node. For the finite game expressed in strategic form, the perfect equilibrium is obtained by a corresponding process of sequential elimination of weakly dominated strategies onto the game matrix: if a strategy is weakly dominated by another without being equivalent to it, it is eliminated; if two strategies are equivalent, we eliminate the one that contains actions present in the strategies that could previously be eliminated directly.

Obviously, all perfect equilibrium states are Nash equilibrium states, which leads us to qualify the concept of perfect equilibrium as a “refinement” of the concept of Nash equilibrium. It is in fact a Nash equilibrium that is robust with regard to “trembles” which appear during the implementation of actions. In a finite game, as the process of backward induction is constructive, there is always a perfect equilibrium. Moreover, this equilibrium is unique when the game is generic, in other words when the same player never obtains the same utility in two different terminal nodes (or if one player obtains the same utility, the other obtains the same as well). Of course, this does not mean that all games have a Nash equilibrium in pure strategies: the strategic forms obtained from the extensive forms are very particular and, conversely, not all strategic forms can be translated into extensive forms (without uncertainty). In an infinite game, on the other hand, there may be a multiplicity of perfect equilibria.

In the entry game, there are two Nash equilibria. The first is the perfect equilibrium obtained by backward induction: in the second period, if the potential entrant has entered the market, it is in the monopolist’s interest to be pacific; in the first period, knowing this, it is in the potential entrant’s
interest to enter the market. The second Nash equilibrium corresponds to an incredible threat; the monopolist threatens to be aggressive if the potential entrant enters the market and the latter chooses not to enter; in fact, this threat would not be put into action if the potential entrant really did enter the market. This illustrates the fact that a good threat is a threat that does not have to be put into action; if it is executed, this means that it has failed. In the ultimatum game, the Nash equilibria are all the possible shares. But the perfect equilibrium is unique: the first player proposes to keep nearly all the cake for himself, as he expects that it is in the interest of the second player to accept. This equilibrium appears to be paradoxical and it is in fact refuted by experimental game theory. In the centipede game, there is one unique Nash equilibrium, which is also perfect: the first player stops the game on the first move. This equilibrium also appears to be paradoxical insofar as the outcomes show Pareto improvement as the game continues.

0.4.5 Folk theorems

By extension of dynamic games under certainty, one can consider the special case of “repeated games”, obtained when the same basic game (static or dynamic) is repeated from one period to another, with a finite or infinite horizon. Each player still possesses perfect information about the structure of the game and past actions taken, but not about the moves played by the other players in the present period. The intertemporal utility function used is generally a discounted sum of the utilities obtained by the player at each repetition of the basic game, using a discount factor \( \delta \):

\[
U(t) = \sum \delta^t U_i(s_i(t), s_j(t)).
\]

It is usually standardized, to make it comparable to the utility function of the basic game, by dividing it by a factor \( \frac{1}{1-\delta} \). The pure strategy of a player is still defined as the action chosen at any period for any history of the game. The behavioral strategy consists in defining a probability distribution on the present possible actions conditional to the past history. Because of the huge increase in the combinations of strategies accessible to the players, the feasible outcomes are also greatly multiplied. Thus, for a discount factor of 1 and an infinite horizon, the feasible outcomes in a repeated game describe the convex envelope of feasible outcomes in the basic game with pure strategies (or mixed strategies).

In repeated games, it is possible to characterize Nash equilibria and perfect equilibria by means of “folk theorems”, so called because they were demonstrated simultaneously by several authors. The object of this characterization is not the equilibrium strategies but the equilibrium outcomes. An outcome is said to be feasible when there exists a couple of strategies that leads to this outcome. An outcome is said to be individually rational if it strictly provides each player with the minimal utility that the other player can (when playing first) impose on him if he is deliberately hostile.
It can be shown that all outcomes of dynamic Nash equilibria are not only feasible but also individually rational. The aim of folk theorems is to examine whether this property holds true in the other direction. They are largely based on the threats (and promises) made by a player concerning the consequences for the other player of deviating from his equilibrium path.

On one hand, for repeated games with infinite horizon, the number of Nash or even of perfect equilibria is considerably multiplied in comparison with the basic game. It can be demonstrated that, for a sufficiently large discount factor, all feasible and admissible outcomes of the game can be obtained (up to $\varepsilon$) by a dynamic Nash equilibrium (exactly, when $\delta = 1$). A similar result can be obtained for perfect equilibria. On the other hand, for repeated games with finite horizon, the number of Nash or perfect equilibria is drastically reduced. If all Nash equilibria of the basic game provide exactly individually rational utility and not more, only the repetition of such a basic equilibrium is a perfect equilibrium of the repeated game. Likewise, if one sole Nash equilibrium exists, the only dynamic perfect equilibrium is the repetition of this Nash equilibrium. However, if the basic game has several Nash equilibria that are not strictly individually rational, the number of possible equilibria in the repeated game is once again greatly multiplied. Thus, under various different assumptions, any feasible and individually rational outcome is once again obtained (up to $\varepsilon$) by a dynamic Nash equilibrium or a perfect equilibrium, if the game horizon is sufficiently long.

In the case of the repeated prisoner’s dilemma, if the game horizon is infinite, all feasible outcomes that give each player a minimum utility of 1 can be obtained by a dynamic Nash or perfect equilibrium, if the discount factor $\delta$ is high enough. In particular, it is possible to obtain the symmetrical Pareto-optimal outcome if $\delta > \frac{1}{2}$, the players refusing to confess in each period of the game. The dynamic Nash equilibrium is supported by the threat of the “grim” strategy, which consists in not confessing in the first round, then continuing not to confess as long as the other player does not confess, but switching to unforgiving, constant confession if the other player should confess just once. The perfect equilibrium is supported by the threat of the “tit for tat” strategy, which consists in not confessing in the first round, then confessing if the other player confessed in the previous round and not confessing if the other player did not confess in the previous round. If the game horizon is finite, only the Nash outcome of the repeated basic game can be obtained, with the two players confessing in each round. For in the last round, it is in the interest of both players to confess; in the penultimate round, the same is true as the action chosen here will have no effect on the other player’s action in the last round; in the anti-penultimate round, the same is therefore true and so on.
0.5 Static games with incomplete information

0.5.1 Behavior assumptions

Let us now return to the situation of a static game in which the two players make simultaneous moves. However, assume that for each player, the choice determinants of the other players are partially unknown; the player is then confronted with “structural uncertainty” or “incomplete information”. On the other hand, each player is assumed to know the determinants of his own choice; the player is then said to be in a situation of “asymmetric information”. A player’s uncertainty can firstly concern the opportunities of the other player, if the latter’s means of action are not well known. But this is reduced to uncertainty about preferences by considering that an action that is not envisaged is envisaged with a utility equal to $-\infty$. A player’s uncertainty then concerns the preferences of the other player if the latter’s utility function is not well known. This in turn can be reduced to uncertainty about beliefs, insofar as each player acts in accordance with a belief about his preferences. Finally, uncertainty concerns the beliefs of the other player and in particular his beliefs about the structure of the game (essentially the players’ choice determinants).

To begin with, the structural uncertainty of player $i$ is “parametrized” by assuming that he considers his opponent to be of a certain type $t_j$, drawn from a set $T_j$ of possible types. The type of the opponent thus summarizes the choice determinants that guide his decisions. A difficulty arises in the case of beliefs, for each player has primary uncertainty about the physical elements of the system. The other player then has secondary uncertainty about the uncertainty of his opponent, which involves the players in a system of crossed beliefs about their respective uncertainties. The type is then nothing other than the hierarchy of crossed beliefs of a player concerning the physical elements of the game, and this hierarchy can be constructed in a coherent way. The set of belief hierarchies of all the players constitutes a “possible world” of a psycho-physical nature. Within a hierarchy, each player is generally expected to know his own type.

It is moreover assumed that each player attributes probabilities concerning the other player’s type, i.e. defines a probability distribution on $T_j$. The belief hierarchy is then formed of a probability distribution on the physical states of the system, a probability distribution on the other player’s probability distribution, and so on. The probabilistic belief hierarchies are “coherent” when a prior probability distribution exists on the whole set of types and is shared by all the players; this is denoted $p(t_i, t_j)$. Such a prior probability distribution is justified by the “Harsanyi doctrine”, according to which the players’ current beliefs only differ in relation to private messages received by them, but are founded on initial common beliefs. Using this common probability, each player calculates the probability of the other’s type, conditionally on his own (known) type: $p_i(t_j/t_i) = p(t_i, t_j)/p(t_i)$. The
players’ types are often correlated, particularly if they are influenced by a common factor. But they may be independent if the players are genuinely isolated; the prior probability then breaks down into independent probability distributions for each player: \( p(t_i, t_j) = p(t_i) \cdot p(t_j) \).

As for the players’ preferences, they are still expressed by utility functions. For player \( i \), his utility \( U_i(s_i, s_j, t_i, t_j) \) depends both on the actions and the types of the two players. The utility of a player depends on his own type if the type concerns precisely the preferences of this player; it depends on the other player’s type if this type has a bearing on, for example, the information of the other player that is liable to influence the first player’s utility. For a finite game, the strategic form is conserved, but the bi-matrix of the game is multiplied by the number of possible types. Because of the probabilistic form attributed to the other player’s type, each player is led to calculate his expected utility, which again imposes the attribution of a cardinal structure to the basic utility. One special class of Bayesian games is that of “Bayesian direct-revelation games” in which the action of the players consists in announcing (truthfully or not) their assumed type, the utility depending on both the type announced and the real type of each player.

### 0.5.2 Elementary games

For the battle of the sexes, each player is now assumed to be uncertain about the other’s preferences. If the other player is of an “altruistic” type, he will value going to his preferred show at 1 and being together at 2. If he is of an “egoistic” type, he will value going to his preferred show at 2 and being together at 1. These two types have prior probability \( p \) and \( 1-p \) respectively. If the two players are altruistic, we obtain the usual matrix; if the two players are egoistic, we obtain the matrix shown below. Two other matrices can be obtained by mixing these two extreme matrices.

<table>
<thead>
<tr>
<th>wife</th>
<th>ballet</th>
<th>boxing</th>
</tr>
</thead>
<tbody>
<tr>
<td>boxing</td>
<td>(1,3)</td>
<td>(2,2)</td>
</tr>
<tr>
<td>ballet</td>
<td>(0,0)</td>
<td>(3,1)</td>
</tr>
</tbody>
</table>

The “lemon game” provides another interesting example. A seller (player 1) decides to sell or not to sell a used car to a buyer (player 2) who decides to buy or not to buy it; the exchange takes place if both players agree. The car can be of good or bad quality. This quality is known to the seller but not to the buyer, who attributes an equiprobability to the car being of good or bad quality. The seller gives a good car a utility of 6 and a bad car a utility of 0; the buyer gives a good car a utility of 10 and a bad car a utility of 0; the price fixed for the exchange (if it takes place) is 4. Here, the car quality plays the role of the type (of the seller) and the matrices for a good and a bad quality car are shown below (taking the absence of exchange as the reference point).
In the duopoly, each firm is endowed with uncertainty concerning different elements of the game structure. The uncertainty may concern, for example, the global demand function; if the latter is linear of the form $a-bq$, each firm can be uncertain about the parameters $a$ or $b$. More often, the uncertainty concerns the production cost function of the other firm; if scale returns are constant, this uncertainty applies to the unit cost $c_i$ of the other firm (each firm knows its own unit cost). In a standard configuration, the first firm is considered to have a marginal cost $c_1$ that is known by both firms; the second firm has a marginal cost that can have a low value $c_2$ or a high value $c_2^*$ for the first firm, but which is known to the second.

### 0.5.3 Strategies

The choices of the players still bear on strategies, but in an extended sense. A strategy of player $i$ is none other than the action of this player, conditional on his type: $s_i(t_i)$. As each player knows his own type, the strategy determines univocally the action that he will take. For his opponent, on the other hand, the strategy determines the action the player should take according to the type he is assumed to endorse. The utility function of player $i$ thus depends on his type (and the other’s type) in two ways: firstly through his strategy and secondly directly: $U_i(s_i(t_i), s_j(t_j), t_i, t_j)$. Of course, one could consider a player implementing mixed strategies, but this case is rarely envisaged. A “conjecture” is still one player’s expectation of the other’s strategy. In a Bayesian direct-revelation game, the strategy of a player describes the type announced by the player in relation to his real type.

In the battle of the sexes, there are four conditional strategies for each player, which appear more or less natural: always go to the boxing, always go to the ballet, go to one’s preferred show if one is egoist and to the other if one is altruist, go to one’s preferred show if one is altruist and to the other if one is egoist. In the lemon game, there are two conditional strategies for the buyer (to buy or not to buy) and four conditional strategies for the seller (always sell, never sell, sell only if the quality is good, sell only if the quality is bad). In the duopoly game, the conditional strategy of the first firm is embodied in its level of production $q_1$ and the conditional strategy of the second firm consists in producing $q_2$ if its cost is low and $q_2^*$ if its cost is high.

### 0.5.4 Bayesian equilibrium

The equilibrium concept introduced here is that of “Bayesian equilibrium”. In fact, this is simply the concept of Nash equilibrium applied to the game
extended to type-conditional strategies. At equilibrium, the conditional strategy of a player is the best reply to the conditional strategy of the other player. Finite games (with finite types) always have a Bayesian equilibrium, possibly with mixed strategies. A Bayesian direct-revelation game is said to be “revealing” if the Bayesian equilibrium associated with it is such that each player truthfully announces his real type: \( s_i * (t_i) = t_i \). The “revelation principle” asserts that with every Bayesian game, it is possible to associate a revealing Bayesian direct-revelation game (with adapted utility functions) whose Bayesian equilibrium leads to the same outcome as that of the primitive game. Therefore, if simply interested in utilities (and not strategies), one can restrict his attention to revealing direct-revelation games.

In the extended battle of the sexes, one can firstly observe that an egoistic player has a dominant action, namely to go to his or her preferred show. The only conditional strategies that subsist are therefore the “uniform strategy” by which a player always goes to his/her preferred show and the “alternate strategy” by which a player goes to his/her preferred show if he is egoistic and to the preferred show of his/her partner if he/she is altruistic. The first strategy cannot lead to a Bayesian equilibrium because, for an altruist, the best reply to the fact that the other player is going to his preferred show is to follow him or her. The second strategy, on the other hand, leads to a symmetric Bayesian equilibrium, but only if \( p \leq \frac{1}{4} \). In this equilibrium, two altruistic players end up, paradoxically, at each other’s preferred shows, two egoistic players each go to their preferred show, whereas if one player is egoistic and the other altruistic, the egoist will go to his or her preferred show, accompanied by the other player. This is revealing, as the action of the player enables one to deduce his or her type.

In the lemon game, the uncertain buyer implicitly considers a matrix which is the average of the two matrices with known types and appears below. In this matrix, the equilibrium is obtained when the seller sells and the buyer buys. It is non-revealing as the exchange takes place in all cases. In the case of perfect information, on the other hand, there is no exchange at the price fixed: the buyer only buys if the car is of good quality, in which case the seller does not sell; the seller only sells when the car is of bad quality, in which case the buyer does not buy.

<table>
<thead>
<tr>
<th>1/2</th>
<th>buy</th>
<th>not buy</th>
</tr>
</thead>
<tbody>
<tr>
<td>sell</td>
<td>(1,1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>not sell</td>
<td>(0,0)</td>
<td>(0,0)</td>
</tr>
</tbody>
</table>

In the duopoly game, the calculation of a Bayesian equilibrium is more complicated. It ends up with the following quantities produced:

- the first firm produces a quantity
  \( q_1 = \frac{[a - 2c_1 + (c_1^2 + c_2^2)/2]}{3b} \)
- the second firm produces
  \( q_2 = \frac{[a + c_1 - (7c_1^2 + c_2^2)/4]}{3b} \)
  \( q_0^2 = \frac{[a + c_1 - (7c_1^2 + c_2^2)/4]}{3b} \)
From this, one can deduce the equilibrium price $p'$ and the profit of the firms $\Pi'$ if the second firm has actually low cost (and symmetrically for high cost):

- the first firm obtains a profit $\Pi_1' = (1/9b)[a - 2c_1 + (5c_2' - c_0')/4][a - 2c_1 + (c_2' + c_0')/2]$
- the second firm obtains a profit $\Pi_2' = (1/9b)[a + c_1 - (7c_2' + c_0')/4]^2$

0.5.5 Value of information

A Bayesian game, which assumes structural uncertainty of the players, leads naturally to the concept of the value of information. Indeed, one can compare, for the players, Bayesian equilibrium states relative to different information structures for their respective types. In particular, one can compare the equilibrium situations before and after reception of a given message. The value of information for a player is simply the difference between the utilities he obtains before and after reception. As concerns the strength of the message, it can be defined as perfect (it completely reveals a player’s type) or imperfect (it only enables a player’s type to be defined in set-theoretic or probabilistic terms). As concerns the target of the message, it can be public (both players receive the message), private (only one player receives the message, the other player knowing that he has received a message of a certain type without knowing its content) or even secret (only one player receives the message and the other player is unaware that he has received it).

Easy calculations show that, for a variety of games, the value of (perfect) information can just as well be negative as positive. However, it can be proved that the value of secret information is positive in any game for the player receiving it; it recalls the situation of choice against nature, where the value of information is still always positive. Likewise, in zero-sum games, the value of public information is positive for some well-defined categories of message. On the other hand, in the case of a (perfect) private message, it can be shown that all the different cases are possible, depending on the games under consideration. The two players can both find themselves better off, the player receiving the message can be better off and the other player worse off, the receiver can be worse off and the other player better off or both players can end up worse off. This situation may seem paradoxical insofar as a player receiving a message of negative value may be tempted to ignore it. In fact, the other player knows that he has received the message and takes this knowledge into account, so that the first player could find himself in an even worse situation.

In the battle of the sexes under uncertainty, the average utility of each player at equilibrium (if $p \leq 1/4$) is $(3 + 2p)(1 - p)$. In the case of certainty about types, if the altruists choose one or the other show with equiprobability, the average utility is increased by $5/2p^2$. The value of public information
concerning the players’ types is therefore positive for both players. In the lemon game, the utility of each player under uncertainty is -1 whereas it is 0 under certainty. The value of the information communicated to the buyer concerning the quality of the car is negative for both the seller and the buyer. In the duopoly game, two cases appear, depending on whether the cost of the first firm is high or low. If it is low, the transmission to the first firm of the information about the second firm’s cost has a negative value for the first firm and a positive value for the second. If the cost is high, the transmission of the same information has a positive value for both firms.

0.6 Dynamic games with imperfect information

0.6.1 Behavior assumptions

Consider now a dynamic context into which some uncertainty is introduced. It is a new form of uncertainty, bearing on the past actions of the players; the player is thus confronted with “factual uncertainty” or “imperfect information”. Here again, it is often assumed that each player has a perfect memory of his own past actions; the player faces a situation of “asymmetric information”. In fact, imperfect information arises not only when a player cannot observe his opponent’s moves, but also when he ignores past events. As in the theory of individual decision, an additional agent called nature is introduced. Nature is liable to clothe states through a random mechanical process, and these states are (provisionally) unknown to the players. It is generally assumed, however, that the players know a prior probability distribution on the states of nature. As the game develops, nature may make successive moves, the corresponding states of nature being correlated or not.

The extensive form of the game shaped as a tree is conserved, but uncertainty is represented by “information sets” on the tree. For a player whose turn it is to move, the nodes between which he cannot distinguish (because they correspond to indiscernible past events) are grouped together in one information set. If the player has a perfect memory, the information sets are formed of nodes situated at the same level of the tree. At each node of an information set, of course, the possible action sets of a player must be identical (otherwise he could partially infer from them the node on which he is currently situated). The information sets of one player form a partition of the nodes where he has to move. The situation of perfect information is obtained when each information set has exactly one node. As for preferences, they are still expressed by the outcomes obtained at the terminal nodes of the tree, i.e. for all paths consisting of actions of players and states of nature, but of course the corresponding utilities only concern the players.

The representation of a game in extensive form with imperfect informa-
tion is very widespread. All static games can be expressed in this form, by assuming that one of the players plays first, the other player then playing without knowing what move the first player has made. Of course, there are several ways of proceeding, with each player being represented as playing first in each period. A fortiori, all repeated games can be represented in this form. All static games with incomplete information can also be expressed in this form, provided that the probability distribution on types is coherent. The attribution of types to the players is performed by nature, which plays the first move and which carries out a draw with the help of a a priori probability distribution attributed to the types of players.

0.6.2 Elementary games

One typical game is “simplified poker”. At the start of the game, each player pays a sum of 1 into the pot; nature gives the first player a card, high or low with equiprobability, which the second player cannot see. Having looked at his card, the first player can either fold, in which case the second player wins the pot, or raise by paying 2 more into the pot. If he raises, the second player can either fold, in which case the first player wins the pot, or call to see the card by paying 2 into the pot himself. If the second player calls, the first player wins the pot if the card is high and the second wins if it is low.

\[
\begin{array}{c|c|c|c|c|c}
C & R & H & L & R & C \\
\hline
2 & 1 & N & 1 & 2 & 2 \\
\hline
F & F & F & F & F & F \\
3 (-3) & 1 (-1) & (-1,1) & (-1,1) & 1 (-1) & 1 (-3,3)
\end{array}
\]

Another example is a variant of the entry game. We now assume that the potential entrant can choose between not entering, entering softly or entering forcefully; in this last case he wins the whole market if the monopolist is pacific. The monopolist cannot distinguish between forceful and soft entry, although he obtains different utilities in each case.

\[
\begin{array}{c|c|c|c|c}
A & E & E & A \\
\hline
2 & 1 & 2 & 2 \\
\hline
P & P & P & P \\
-1 (-1) & 2 (0) & 0 (2) & 1 (1) & -1 (-1)
\end{array}
\]

0.6.3 Strategies

The pure strategy of a player is now defined as the move made by the player in every information set (i.e. the same move at each node of the information
As mixed strategy, one can again define a random strategy (probability distribution on pure strategies) or a behavioral strategy (probability distribution in all information sets). Nature, in particular, uses a behavioral strategy, known by the players (contrary to the precise state). If nature only plays once, it can be considered without restriction as playing first. When a static game with incomplete information is reduced to a dynamic game with imperfect information, conditional strategies become simply ordinary strategies relative to the information set initially induced by nature. If nature does not intervene, precise utilities can be associated with every couple of strategies. If it does intervene, expected utilities in relation to the states of nature can be associated with every couple of strategies. The extensive form of the game can then be expressed in strategic form with the help of the strategies and their corresponding utilities. However, two extensive forms giving rise to the same strategic form do not necessarily induce the same strategic predictions.

In the game of poker, the first player has four strategies, each strategy indicating what he does depending on his card; for instance, FR means that he folds with a high card and raises with a low one; the second player can only choose between folding or calling. In the entry game, the strategies coincide with the actions. The strategic forms of the two games are then as follows:

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & C & F & P & A \\
\hline
RR & (0,0) & (1,-1) & -E & (0,2) \\
RF & (1,-1) & (0,0) & Es & (1,1) \\
FR & (-2,2) & (0,0) & Ef & (2,0) \\
FF & (-1,1) & (-1,1) & & \\
\hline
\end{array}
\]

0.6.4 Nash equilibrium and perfect equilibrium

The most common equilibrium in dynamic games with imperfect information is still the Nash equilibrium. It is obtained by considering the normal form of the game, constructed from the strategies. The perfect equilibrium of a dynamic game with imperfect information can also still be defined. But it is no longer possible to consider the subgames of the initial game and they are replaced by the “proper subgames”, i.e. the subgames that do not cut any information sets (and which therefore begin with a ‘singleton’ node). A perfect equilibrium is then a profile of strategies which induces a Nash equilibrium in all proper subgames of the game. In particular, if the game is finite, the backward induction procedure applies to the nested proper subgames: beginning at the game horizon, each proper subgame is replaced one by one by its Nash equilibrium outcomes. However, perfect equilibrium eliminates a lot less Nash equilibria under uncertainty than under certainty; it is even possible that strongly dominated strategies are conserved if they are not inserted into a proper subgame.
In the game of poker, the last two strategies (where the player folds with a high card) are strongly dominated. The Nash equilibrium of the residual game is an equilibrium in mixed strategies. The first player always raises if the card is high and raises one out of two times if the card is low. The second player folds one out of two times. This equilibrium is perfect because no proper subgames exist outside of the complete game. It can be observed that in this game, player 1 uses a “deterrent” form of bluff (a “strong” raise when he has a low card, to induce his opponent to fold). However, the “incentive” form of bluff is not used in this game (a “weak” raise when he has a high card, to encourage his opponent to call). In the entry game, the strategy that consists in entering softly is weakly dominated. There are still two Nash equilibria. In the first, the potential entrant enters forcefully and the monopolist is pacific. In the second, the potential entrant does not enter because the monopolist threatens to be aggressive. However, whereas the first equilibrium is still perfect, the second is also perfect because no proper subgames exist outside of the game itself. Nevertheless, the threat of aggression remains incredible because if the potential entrant enters, being aggressive is (strongly) dominated by being pacific.

0.6.5 Perfect Bayesian equilibrium

In order to consider non-proper subgames, one explicitly introduces players’ beliefs about their corresponding information sets. These probabilistic beliefs specify the node of the information set on which the player believes himself to be situated, or, equivalently, which game history has led him to this information set. More precisely, in an information set $h$ of player $i$, denote $\mu_i^h$ a probability distribution on the constituent nodes. Each player $i$ is thus defined by a pair $(\mu_i, \sigma_i)$ formed of a belief system $\mu_i$ and a behavioral strategy $\sigma_i$, these two elements being connected by two conditions.

The behavioral strategy $\sigma_i$ of player $i$ is “sequentially rational” if, in all information sets $h$ (whether or not these are reached by the strategies of the players), it is the best reply to the behavioral strategy $\sigma_j$ of the other player, taking into account present and future beliefs $\mu_i^h$. The belief system $\mu_i$ is “weakly coherent” if, in all information sets $h$ reached by the strategies of the players (but not necessarily the others), the belief $\mu_i^h$ is deduced from these behavioral strategies by Bayes’ rule.

A “perfect Bayesian equilibrium” is nothing other than a pair $(\mu, \sigma)$ that obeys the above two conditions. It results from a fixed point of a meta-loop adapting the beliefs and behavioral strategies of all the players to each other. This loop is superimposed on - and even replaces - the classic loop connecting simply the players’ strategies to each other. It guarantees both the cognitive rationality of the players (the appropriateness of their beliefs in relation to the information) and the instrumental rationality of the players (the appropriateness of their actions in relation to their preferences). In general, multiple Bayesian equilibrium states exist for all games; many
of them share the same strategies and only differ in terms of their beliefs. However, no systematic algorithm exists for calculating perfect Bayesian equilibria; the modelizer must guess the possible equilibrium states and then verify his or her intuition.

In the game of poker, the unique Nash equilibrium is a perfect Bayesian equilibrium. It is supported by the belief of the second player that he will find himself, in his information set, with a high card with a probability of 2/3 and with a low card with a probability of 1/3. In the modified entry game, the perfect equilibrium in which the potential entrant is deterred from entry by the monopolist’s threat of aggression is not sequentially rational, whatever the monopolist’s belief, because the action of being aggressive is strongly dominated. The perfect equilibrium in which the potential entrant enters and the monopolist is pacific, on the other hand, is a perfect Bayesian equilibrium, supported by the monopolist’s belief that if the potential entrant enters, he will do so forcefully. In the battle of the sexes rewritten in extensive form, the three equilibria (two in pure strategies, one in mixed strategies) are perfect Bayesian equilibria.

Reconsider now a repeated game without updating, with finite horizon T, but this time under uncertainty (concerning types). Once again, by virtue of an extended folk theorem, the presence of uncertainty greatly multiplies the number of equilibria (as it does in a game with infinite horizon). More precisely, it is proved that one can find an irrationality of the players such that, if it is envisaged with sufficient probability by all players for their opponent, any feasible and individually rational outcome of the game can be approached up to ε by a perfect Bayesian equilibrium. In the repeated prisoner’s dilemma, each player assumes that the other is of a rational type with probability ε and that he is of irrational type who plays the “tit-for-tat” strategy systematically with probability 1 - ε. Consequently, if each player is truly rational and if the game horizon is sufficiently far off, the players cooperate during the first T-k periods (where k only depends on ε) and cease to cooperate for the last k periods.

0.7 References

Friedman, J. (1990): Game theory with applications to economics, MIT Press
Guerrien, B. (1993): La théorie des jeux, Economica
Luce, R., Raiffa, H. (1957): Games and decisions, Wiley