Charity Auctions for the Happy Few*

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Abstract

Recent literature has shown that all-pay auctions raise more money for charity than winner-pay auctions. We demonstrate that the first-price and second-price winner-pay auctions outperform the first-price and second-price all-pay auction when bidders are sufficiently asymmetric. To prove it, we consider a framework with complete information. Complete information is realistic and corresponds to events that occur, for instance, in a local service club (such as a voluntary organization) or in a show business dinner.

KEYWORDS: All-pay auctions, charity, complete information, externalities

JEL CLASSIFICATION: D44, D62, D64

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1 Introduction

More and more voluntary organizations wish to raise money for charity purposes through a partnership with firms. While Charity auctions have been held in the United States for many years now, in China this phenomenon has emerged recently and is in strong progress.[1] In this kinds of auction, an object is sold (for example a key case with zero value or an item given by a luxury brand). The proceeds then go to charity. Most of these auctions are planned and organized in charity dinners where only wealthy or famous people can participate. Beyond the item value, the valuations of potential bidders depend on their interest in this voluntary organization (their altruism or philanthropy) and on the degree to which they show some kind of conformism "to be seen as the most wealthy and generous". For instance, in China's traditional society, charity auctions were not organized. The participants preferred to keep a low profile about their bids. However, time has changed: the rich and famous now display their wealth through their involvement in charity auctions. According to the *Beijing Review*:

> *With the development of society, more rich people are emerging. They have their own lifestyle [...] Some day, behind the rich lifestyle, people will find that it is only by offering their love and generosity that they can realize their true class.*

Thus, through charity auctions, potential bidders can build their position in their social class. Everybody wishes, independently of the winner's identity, to raise the highest revenue. Potential bidders make a trade-off between giving money for the fundraising and keeping it for another personal use. Contrary to non-charity auctions though, here the amount paid is "never lost". A wealthy investor, who bought a Dior perfume for 60 000 yuans (about 6 000 euros or 7 700 dollars) – with a reserve price of 20 000 yuans – recently said in the *Beijing Review*:

> *I would never buy perfume for this amount normally, but this time it is for charity. I feel very happy.*

Since, in fact, the money raised is used to finance a charitable purpose, every participant of the charity auction can benefit from it, independently of the winner's identity. More precisely, the money raised by each potential bidder impacts the utility of all participants as they take advantage of an externality of the amount of the money raised for the charity purpose.

Under complete information, these kinds of auctions can be compared to the work of Ettinger (2002) who analyzes a general winner-pay auction framework with financial externalities.[2] These externalities do not depend on the winner's identity and can be applied to charity auctions where only the winner pays.[3] Moreover, he shows that there is no "revenue equivalence" with these externalities. Maasland and Onderstal (2007) investigate winner-pay auctions with this kind of linear externalities in an independent private signals model. Their paper can also be applied to charity. They find similar qualitative predictions as Ettinger (2002): the second-price winner-pay auction outperforms[4] the first-price winner-pay auction. In their recent paper, Goeree et al.

[1] For example, in 2004, at the Formula One Grand Prix opening dinner party in Shanghai (China), *an auction was held of racing suits and crash helmets used by famous racing drivers* (*Beijing Review*, 2005).

[2] To the best of our knowledge, Ettinger (2002) is the only one to consider general externalities which could be non-linear.

[3] Actually, Ettinger (2002) investigates a framework with two kind of externalities. One is independent of the winner's identity and the other depends on the winner's identity.

[4] In the following, *outperform* means *generate higher revenue*.

(2005) analyze charity auctions in the symmetric independent private values model. They show that, given the externality, all-pay auctions raise more money for charity than winner-pay auctions (second-price outperforms first-price) and lotteries. In particular, they determine that the optimal fundraising mechanism is the lowest-price all-pay auction with an entry fee and a reserve price. Engers and McManus (2007) find closed results to Goeree et al. (2005).[5] Contrary to Goeree et al. (2005), a psychological effect comes into play: the winner benefits from a higher externality with her own bid, the others' bids having a lower effect on him. In their setting, as in Goeree et al. (2005), first-price all-pay auctions and second-price winner-pay auctions are better to raise money than first-price winner-pay auctions. Moreover, first-price all-pay auctions outperform each winner-pay auction only for a sufficiently high number of bidders. Additionally, Engers and McManus (2007) show that there are many optimal charity auctions, among them a first-price winner-pay auction with suitable fees and cancelling threat, for example.

The predictions of Goeree et al. (2005) and Engers and McManus (2007) have been tested experimentally with contradictory results. Onderstal and Schram (2009) have tested Goeree et al.'s (2005) results experimentally in the laboratory. They are the first to conduct a lab experiment for charity auctions in an independent private value setting. Their results are closed to the theoretical predictions: in charity auctions, the first-price all-pay auction raises higher revenues than other mechanisms (first-price winner-pay auction and lotteries). Carpenter et al. (2008) have tested the predictions of Engers and McManus (2007) and Goeree et al. (2005) in a field experiment. Similar objects are sold in four American pre-schools through three different mechanisms which are the first-price all-pay auction, and the first-price and second-price winner-pay auctions. They study the determinants of the bidders' behavior and the revenue raised. Contrary to the theoretical predictions, first-price all-pay auctions do not produce higher revenues than the winner-pay auctions. Therefore, if auction theory applied to charity is confirmed in the laboratory, this is not the case in the field. The main explanation for the gap between theory and field experiment could be a non-participation effect, due to the unfamiliarity with these mechanisms and their complexity: the participants did not know the all-pay design and few took part in second-price auctions on the Internet.

The purpose of this paper is to determine whether or not all-pay auctions can raise higher revenue for charity than winner-pay auctions when bidders are asymmetric. We consider a complete information framework. As already mentioned, many charity auctions are conducted among rich people during charity dinners. These events could occur in a local service club (like the Rotary Club[6] or another type of voluntary organization) or during a show business dinner. Potential bidders are acquaintances or know one another well. Consequently, a complete information environment is well suited for these kinds of situation.

We analyze all-pay auctions for charity as a mechanism. This approach relies on a general model which can be applied to both first and second-price all-pay auctions. In our framework, the externalities are such that every bidder takes as much advantage (obtains as much utility)

[5] Besides, Engers and McManus (2007) differentiate situations in which the auctioneer can or cannot threaten to cancel the auction, which change their results.

[6] *The Rotary Club is a worldwide organization of business and professional leaders that provides humanitarian services, encourages high ethical standards in all vocations, and helps build goodwill and peace in the world.* There are about 32 000 clubs in 200 countries and geographical areas and 1,000 clubs in France like Paris, but also in small town like Niort. http://www.rotary.org/

of her own bid as of her rival's bid. Additionally, we define bidder i's adjusted-value as the ratio of her value for the item sold to the fraction of her payment which she perceives as a cost given her altruism for the charity purpose. Then, we arrange bidders in such a way that the adjusted-values and the valuations are ranked in the same order. We discuss this ranking and its consequences.

We characterize the first-price all-pay auction equilibrium and compute the expected revenue; but there is no pure strategy Nash equilibrium. As in a case without externalities, only the two bidders with the highest adjusted-values are active. Moreover, we establish the existence of a Nash equilibrium in mixed strategies with non-linear externalities.

The equilibrium in the second-price all-pay auction is also characterized and the expected revenue computed. Then, we compare our results to Ettinger (2002) who analyzes winner-pay auctions with externalities that do not depend on the identity of the winner and which could be applied to charity auctions.

The revenue of the all-pay auctions can be dominated by the revenue of the winner-pay auctions contrary to the results of Goeree et al. (2005). Actually, above a certain threshold of asymmetry in the bidders' valuations, winner-pay auctions raise more money for charity than the all-pay auctions. Our result can also be related to the work of Carpenter et al. (2008). Indeed, their results could be due to a strong asymmetry between bidders. Moreover, we reexamine our result by an analysis of the bidders' altruism.

2 The model

We describe all-pay auctions for charity as mechanisms. This approach relies on a general model which can be applied to both first and second-price all-pay auctions.[7]

In a charity dinner, an indivisible object (or prize) is sold through an all-pay auction. This prize is allocated to one of the potential bidders $N = \{1, ..., n\}$ contingent upon their bids $\boldsymbol{x} = (x_1, ..., x_n) \in \mathbb{R}^n_+$. As the bidders usually meet each other in these kinds of events, the willingness to pay and the valuation ranking of each bidder, $v_1 > v_2 > ... > v_n$, are common knowledge. An all-pay auction is a pair $(\boldsymbol{a}, \boldsymbol{t})$, $\boldsymbol{a}$ being the allocation rule and $\boldsymbol{t}$ the payment rule.

Allocation Rule. The allocation rule $\boldsymbol{a} = (a_1, ..., a_n) : \mathbb{R}^n_+ \longrightarrow [0,1]^n$ is such that the winner i gets the object if and only if $a_i(\boldsymbol{x}) = 1$ given the bids and $\sum_{i=1}^n a_i(\boldsymbol{x}) = 1$ for all $\boldsymbol{x}$. The object is allocated to the highest bidder such that

$$\begin{cases} a_i(\boldsymbol{x}) = \frac{1}{\#Q(\boldsymbol{x})} \text{ if } i \in Q(\boldsymbol{x}) \\ a_i(\boldsymbol{x}) = 0 \text{ otherwise} \end{cases}$$

where $Q(\boldsymbol{x}) := \{j | j = \arg\max\{x_k, k \in N\}\}$ is the collection of the highest bids.

Payment Rule. The payment rule $\boldsymbol{t} = (t_1, ..., t_n) : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n_+$ represents for each bidder i her transfer $t_i(\boldsymbol{x})$ to the charity organization for the vector of bids $\boldsymbol{x}$. This payment rule is

[7] Vartiainen (2007) uses this kind of general layout for a non-charity framework. Our approach is different. Indeed, in our case, every bidder takes as much advantage of her own bid as of her rival's bid thanks to introduction of the externalities.

contingent upon the all-pay design. In a first-price all-pay auction, each bidder pays her own bid

$$t_i(\boldsymbol{x}) = x_i \ \ \forall i \in N$$

while in the second-price all-pay auction the winner pays the second highest bid and the losers their own bid

$$\begin{aligned} t_i(\boldsymbol{x}) &= x^{(2)} \text{ if } i \in Q(\boldsymbol{x}) \\ t_i(\boldsymbol{x}) &= x_i \text{ otherwise} \end{aligned}$$

with $x^{(2)}$ the second order statistic of the sample $(x_1, ..., x_n)$.

Bidders wish to raise the maximum of money for charity. Each bidder takes advantage of her own participation in the charity auction and of the others' participations as well. In other words, the money raised by each potential bidder impacts the utility of all the participants including herself. Thus, the bidder's utility function includes an externality which depends on the amount of money raised for the charity purpose. Denote $h_i(\boldsymbol{t}(\boldsymbol{x}))$ the externality from which bidder i benefits.[8] We could also consider the externality as a function with only one argument $\sum_{j=1}^{n} t_j(\boldsymbol{x})$. Indeed, the externality is independent of the winner's identity and only takes into account the amount raised. Except in Section 3.2, we make a linearity assumption regarding the form of the externality like Goeree et al. (2005) and other papers on charity auctions:

$$h_i(\boldsymbol{t}(\boldsymbol{x})) = h_i\left(\sum_{j=1}^{n} t_j(\boldsymbol{x})\right) = \alpha_i \sum_{j=1}^{n} t_j(\boldsymbol{x})$$

where $\alpha_i \geq 0$ is the coefficient of bidder i's altruism for the charity purpose. Thus, bidder i's utility is given by

$$U_i(\boldsymbol{x}) = \tilde{U}_i(a_i, \boldsymbol{t}) = v_i a_i(\boldsymbol{x}) - t_i(\boldsymbol{x}) + \alpha_i \sum_{j=1}^{n} t_j(\boldsymbol{x})$$

The next assumption is useful only for non-linear externalities.

Assumption 1 (A1)**.** *$\tilde{U}_i(a_i, \boldsymbol{t})$ is a continuous and differentiable function in the transfer functions t_j for all j.*

Thus, in the case of non-linear externalities, $h_i(\boldsymbol{t}(\boldsymbol{x}))$ is continuous and differentiable in all of its arguments.

Assumption 2 (A2)**.** $\forall x_i \geq 0$ $\dfrac{\partial \tilde{U}_i}{\partial t_i(\boldsymbol{x})}(a_i, \boldsymbol{t}) < 0$ *equivalent to* $\alpha_i \sum_{j=1}^{n} \dfrac{dt_j(\boldsymbol{x})}{dt_i(\boldsymbol{x})} < 1.$

This assumption means that the bidder has a strict preference to keep one euro for her own use rather than to give it to the charity auction. This is the limit to the bidders' altruism to give money for charity.[9] The limit of the bidders' altruism is affected by the payment rule. Indeed, bidder i's transfer can be a function of her opponents' bids and then do the same transfers. Thus, a change in the payment rule leads to a new limit of the bidders' altruism: in first-price

[8] The vectors $(t_1(y), ..., t_n(y))$ and $(t_1(\boldsymbol{z}), ..., t_n(\boldsymbol{z}))$ are denoted $\boldsymbol{t}(y)$ and $\boldsymbol{t}(\boldsymbol{z})$.

[9] If $\alpha_i \sum_{j=1}^{n} \frac{dt_j(\boldsymbol{x})}{dt_i(\boldsymbol{x})} = 1$ then the bidder is indifferent between giving one euro for charity or investing it in an another activity.

it is $\alpha_i < 1$ while in second-price $\alpha_i < 1/2$.

Denote $F_i(x) \equiv \mathbb{P}(X_i \leq x)$ the cumulative distribution functions such that bidder i decides to submit a bid inferior to x. We denote $F_i(0)$ the probability that bidder i bids 0. When $F_i(0) \neq 0$, bidder i bids zero with a strictly positive probability. When $F_i(0) = 1$, bidder i always bids zero which means that she does not participate in the auction. $F_1, ..., F_n$ can be interpreted as the bidding strategies where the support is $\mathbb{R}_+$. Thus, the expected utility of bidder i is given by:

$$\mathbb{E}U_i(x_i, \boldsymbol{X}_{-i}) = \int_{\mathbb{R}_+^{n-1}} \left(v_i a_i(\boldsymbol{x}) - 1 t_i(\boldsymbol{x}) + \alpha_i \sum_{j=1}^{n} t_j(\boldsymbol{x}) \right) \prod_{j \neq i} dF_j(x_j) \tag{1}$$

$$\begin{aligned} &= v_i \prod_{j \neq i} F_j(x_j) - 1 \int_{\mathbb{R}_+^{n-1}} t_i(\boldsymbol{x}) \prod_{j \neq i} dF_j(x_j) \\ &\quad + \alpha_i \int_{\mathbb{R}_+^{n-1}} \sum_{j=1}^{n} t_j(\boldsymbol{x}) \prod_{j \neq i} dF_j(x_j) \end{aligned} \tag{2}$$

with $\boldsymbol{X}_{-i} = (X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$. From (1) and (2) we can notice that the events $\{\#Q(\boldsymbol{x}) = 1\}$ and $\{\#Q(\boldsymbol{x}) > 1\}$ are disjoint. Thus, when $\#Q(\boldsymbol{x}) > 1$ the value of the integral is zero. Indeed, a tie is a zero measure event.

Let us denote by $\frac{v_i}{1-\alpha_i}$ bidder i's adjusted-value. Bidders i's adjusted-value is defined as the ratio of her value for the item sold and the fraction of her payment which she perceives as a cost given her altruism for the charity purpose. We can observe this adjusted-value in the expected utility with a normalisation by dividing it by $1 - \alpha_i$. As bidders are *ex ante* asymmetric, we arrange them such that $\frac{v_i}{1-\alpha_i}$ decreases with the suffix i and without equality. This is common knowledge. Thus,

$$\frac{v_1}{1-\alpha_1} > \frac{v_2}{1-\alpha_2} > ... > \frac{v_n}{1-\alpha_n}$$

3 First-Price All-Pay Auction

In this section, we study the most popular all-pay auction design, *i.e.* the first-price all-pay auction. Every bidder pays her own bid, but only the one with the highest bid wins the object.

Given assumption $A2$, there is no pure strategy Nash equilibrium. This is a well known result when there is no externality. We only provide a sketch of the proof of this result with two bidders for the first-price all-pay auction with externalities.
Let us assume that $x_i \geq x_j$ and consider some general externalities (not necessarily linear) given by $h_i(x_i, x_j)$. In such a framework, two cases can occur. First, if bidder j can overbid, then her best reply is $x_i + \varepsilon$, for $\varepsilon > 0$ such that $v_j - (x_i + \varepsilon) + h_j(x_i, x_i + \varepsilon) \geq -x_j + h_j(x_i, x_j)$. Hence, it is impossible that $x_i \geq x_j$. Second, if j cannot overbid, then his best reply consists in offering zero since, given assumption $A2$, $h_j(x_i, 0) > -x_j + h_j(x_i, x_j)$. Consequently, i's best reply is to offer $\varepsilon > 0$. As a result, the equilibrium is unstable and there is no pure strategy Nash equilibrium.

3.1 Linear Externalities

As we noticed in the last section, assumption $A2$ implies that $\alpha_i < 1$. If bidder i offers x_i, then j will offer less with probability $F_j(x_i)$ and will offer more with probability $1 - F_j(x_i)$. Whatever the outcome, bidder i benefits from the sum of all bids, including her own. When computing her expected utility, she takes the amount paid by each opponent into account. Bidder i's expected utility with n potential competitors is given by

$$\mathbb{E}U_i(x_i, \boldsymbol{X}_{-i}) = \prod_{j \neq i} F_j(x_i) v_i - (1 - \alpha_i) x_i + \alpha_i \sum_{j \neq i} \mathbb{E}X_j$$

A potential bidder takes part in the auction if for some bids her expected utility is equal to or higher than the externalities she benefits when her bid is zero. Formally, a bidder takes part in the auction if

$$\exists \;\; x \text{ such that } \mathbb{E}U_i(x, \boldsymbol{X}_{-i}) \geq \alpha_i \sum_{j \neq i} \mathbb{E}X_j$$

with $\alpha_i \sum_{j \neq i} \mathbb{E}X_j$ bidder i's expected reservation utility when she takes part in the auction. We call the highest price at which a given bidder is ready to take part in the auction her *indifference price*. i's *indifference price* is denoted by $\tilde{x}_i$ and satisfies $\mathbb{E}U_i(\tilde{x}_i) = \alpha_i \sum_{j \neq i} \mathbb{E}X_j$.

Proposition 1. *There is a unique Nash equilibrium and the mixed strategies are given by*

$$F_1(x) = \frac{1 - \alpha_2}{v_2} x \;\; \forall x \in \left[0, \frac{v_2}{1 - \alpha_2}\right] \;\; \textit{and } F_2(x) = 1 - \frac{1 - \alpha_1}{1 - \alpha_2} \frac{v_2}{v_1} + \frac{1 - \alpha_1}{v_1} x \;\; \forall x \in \left(0, \frac{v_2}{1 - \alpha_2}\right]$$

All other bidders use the pure strategy of bidding zero and do not take part in the auction: $F_j(0) = 1$ *for* $j \in \{3, ..., n\}$*. The expected revenue is given by* $\mathbb{E}R = \frac{1}{2} \frac{v_2}{1 - \alpha_2} \left(\frac{1 - \alpha_1}{1 - \alpha_2} \frac{v_2}{v_1} + 1 \right)$.

Thanks to Hillman and Riley (1989) and Baye et al. (1996), this result is easy to obtain. Dividing bidders' i expected utility by $1 - \alpha_i$ we obtain an affine transformation of the expected utility without externality given by Hillman and Riley (1989) and Baye et al. (1996). Indeed, in our case the adjusted-values $\frac{v_i}{1 - \alpha_i}$ have the same play than the value v_i. Moreover, we have an additional term $\frac{\alpha_i}{1 - \alpha_i} \sum_{j \neq i} \mathbb{E}X_j$ which is constant at the equilibrium. As the result of Baye et al. (1996) is invariant with respect of positive affine transformations of expected utility, the mixed strategies are invariant with respect to dividing by $1 - \alpha_i$ and adding a constant to the expected utility. Then our result follows.

Corollary 1. *All bidders obtain a positive payoff. Indeed, the bidders with the two highest adjusted-values obtain a positive payoff* $U_1^\star = v_1 - \frac{1-\alpha_1}{1-\alpha_2} v_2 + \frac{\alpha_1}{2} \frac{1-\alpha_1}{v_1} \left(\frac{v_2}{1-\alpha_2} \right)^2$ *and* $U_2^\star = \frac{v_2}{2} \frac{\alpha_2}{1-\alpha_2}$ *and their competitors get* $U_i^\star = \frac{\alpha_i}{2} \frac{v_2}{1-\alpha_2} \left(\frac{1-\alpha_1}{v_1} \frac{v_2}{1-\alpha_2} + 1 \right)$ *for* $i \in \{3, ..., n\}$.

Proof. Computations. ■

Contrary to the case with no externality (see Hillman and Riley (1989) and Baye et al. (1996)), the highest bidder's opponents get a positive payoff. This is a consequence of the externalities: bidders benefit from their competitors' behavior.

Remark 1. *Let us assume that the difference between α_1 and α_2 is large enough for bidder 1's adjusted-value to be ranked second such that the two highest adjusted-values are permuted. Then bidder 1 can get a lower payoff than in the case with no externality if and only if her altruism level is lower than $\tilde{\alpha} \equiv 2\frac{v_1 - v_2}{3v_1 - 2v_2}$. We notice that this threshold does not depend on her rival's altruism level, while the changes in the ranking of the adjusted-values is only due to the difference between the players' altruism levels.*

We can notice here that there are two opposite effects. Because of the externalities, the value of one euro that is invested in the auction is less than one euro. Thus, it is possible that the bidders choose more aggressive offers. However, each bidder knows that her competitor is more aggressive and that this will affect one's probability of winning. Given an increase of her competitor's aggressiveness, the bidder's best reply could be to raise or lower her bid.

3.2 Non-Linear Externalities

We extend our result to non-linear externalities. We consider two bidders only, such that the expected utility is given by,

$$\mathbb{E}U_1(x_1, X_2) = F_2(x_1)\left(v_1 + \mathbb{E}_{X_2}(h_1(x_1, X_2)\backslash X_2 \leq x_1) - x_1\right) + (1 - F_2(x_1))(\mathbb{E}_{X_2}(h_1(x_1, X_2)\backslash X_2 \geq x_1) - x_1)$$
$$\mathbb{E}U_2(x_2, X_1) = F_1(x_2)\left(v_2 + \mathbb{E}_{X_1}(h_2(X_1, x_2)\backslash X_1 \leq x_2) - x_2\right) + (1 - F_1(x_2))(\mathbb{E}_{X_1}(h_2(X_1, x_2)\backslash X_1 \geq x_2) - x_2)$$

with $\mathbb{E}_{X_2}(h_1(x_1, X_2)\backslash X_2 \leq x_1) = \begin{cases} \dfrac{1}{F_2(x_1)} \displaystyle\int_0^{x_1} h_1(x_1, x_2) dF_2(x_2) \text{ if } x_1 > 0 \\ 0 \text{ otherwise} \end{cases}$

It can also be written as

$$\begin{cases} \mathbb{E}U_1(x_1, X_2) = F_2(x_1)v_1 - x_1 + \mathbb{E}_{X_2} h_1(x_1, X_2) \\ \mathbb{E}U_2(x_2, X_1) = F_1(x_2)v_2 - x_2 + \mathbb{E}_{X_1} h_2(X_1, x_2) \end{cases}$$

Bidder i takes part in the auction if her expected utility is higher than her reservation utility:

$$\exists \;\; x_i \text{ such that } \mathbb{E}U_i(x_i, X_j) \geq \mathbb{E}_{X_j} h_i(0, X_j)$$

Proposition 2. *Given assumptions $A1 - A2$ the mixed strategy Nash equilibrium exists.*

Proof. See in Appendix. ■

The expected utility's derivative is a Fredholm equation of the second type. The existence of a solution depends on a condition made on the kernel (the kernel being the externality here). Nonetheless, given that the solution is a distribution function defined on a closed and convex set of continuous distribution functions, we are able to show its existence by using Schauder's second theorem without this standard condition. The solution seems to be unique only in very specific cases, as a known result in the literature on Fredholm equations shows.[10]

4 Second-Price All-Pay Auction

In a second-price all-pay auction, the payment rule is as follows: the winner pays the second highest bid and the others pay their own bid. Our purpose is now to determine the bidders'

[10] Kanwal (1971) has written a very complete book on these questions while Ledder (1996) provides a simple method and finds another condition to prove the solution's uniqueness.

strategies and the expected revenue. In the next section, we will compare the rents obtained in first-price and second-price auctions, as well as winner-pay and all-pay auctions. As a result, we will know which of these designs is the best to raise money for charity.

It is not necessary to find each agent's probability distribution's support in order to determine the mixed strategy Nash equilibrium. Actually, we only need to assume that each bidder i's offer, x_i belongs to a strategy space $[0, +\infty)$. For the same reasons as in the first-price auction, the bidders' minimum valuations is zero. As noticed before, assumption $A2$ allows us to write that $\alpha_i < 1/2$.

There are a continuum of pure strategy equilibria as in the situations without externalities. Hendricks et al. (1988) show such equilibria are never subgame perfect in the dynamic version of the auction which is strategically equivalent to the static version.[11] Thus, we focus on the completely mixed strategy equilibria. As the support of the strategies is $\mathbb{R}_+$, strategies are completely mixed. Then, strategies are continuous, atomless and gapless (see Moulin (1986) for more details). In the two bidders case the payment rule leads to $t_1(x) = t_2(x)$. Thus, when a bidder wins she pays her rival's bid. In addition, each bidder benefits from two externalities, one associated with her own bid, and the other associated with her rival's bid. Then the expected utility is given by

$$\mathbb{E}U_i(x_i, \boldsymbol{X}_{-i}) = \int_0^{x_i} (v_i - (1 - 2\alpha_i)x)dF_j(x) - (1 - 2\alpha_i)x_i(1 - F_j(x_i)).$$

Dividing by $1 - 2\alpha_i$, this expected utility is qualitatively equivalent to the one without any externalities (see Vartiainen (2007)). However, it is not the case anymore for n bidders. Indeed, externalities have a different effect depending on whether bidder i is the winner, the second highest bidder or a loser with a bid inferior to the second highest bid. Then the expected utility and the equilibrium are non-intuive and difficult to compute with n bidders. Let us note $G_i(x) = \prod_{j \neq i} F_j(x)$. It follows that the expected utility (2) can be written

$$\begin{aligned}\mathbb{E}U_i(x_i, \boldsymbol{X}_{-i}) = &\int_0^{x_i} (v_i - (1 - \alpha_i)x)dG_i(x) - (1 - \alpha_i)x_i(1 - G_i(x_i)) \\ &+ \alpha_i \sum_{l \neq i} \int_{\mathbb{R}_+} x_l \Big(1 - \mathbb{1}_{x_i \leq x_l} \prod_{k \neq l, i} F_k(x_l)\Big) dF_l(x_l) \\ &+ \alpha_i \sum_{l \neq i} \Big(\int_{\mathbb{R}_+} \int_{x_i}^{x_l} \sum_{k \neq l, i} x_k \prod_{\substack{m \neq i,k,l \\ k \neq l}} F_m(x_k) dF_k(x_k) dF_l(x_l) + x_i \prod_{m \neq i, l} F_m(x_i)(1 - F_l(x_i)) \Big)\end{aligned} \quad (3)$$

The transition from the equation (2) to the equation (3) is explained in the proof of Proposition 3 given in the appendix. The two terms in the first line represent bidder i's payoff depending on whether she wins or loses the auction, given the externality that arises from her own action.

[11] We only give an intuitive argument for the two bidder case. Bidder i's expected utility is given by

$$U_i(x) = \begin{cases} v_i + (2\alpha_i - 1)x_j & \text{if } x_i > x_j \\ \frac{v_i}{2} + (2\alpha_i - 1)x_i & \text{if } x_i = x_j \\ (2\alpha_i - 1)x_i & \text{if } x_i < x_j \end{cases}$$

As before, we note $\tilde{x}_i$ bidder i's *indifference price*, such that $\tilde{x}_1 > \tilde{x}_2$. Let x_i be bidder i's offer. Thus, pure strategy Nash equilibria are

$$(0, \beta_1) \text{ with } \beta_1 \in (\tilde{x}_1, +\infty)$$
$$(\beta_2, 0) \text{ with } \beta_2 \in (\tilde{x}_2, +\infty)$$

and the revenue is zero.

The other lines represent the externalities that come from her competitors' actions (whether they lose or win).

The first of these two lines describes the situation where bidder l ($l \neq i$) loses the auction. In the last line bidder l wins the auction; we distinguish situations where bidder i's bid is the second highest offer from situations where she is not. Each bidder's offer could be the second highest bid and we take account of it (sum operator under the integral). The bidder who makes an offer between bidder i and bidder l's offers puts forward the second highest bid. The other part of the equation gives the amount of money that bidder l has to pay when i offers the second highest bid. Indeed, $\prod_{m \neq i,l} F_m(x_i)(1 - F_l(x_i))$ is the probability that every bidder except l makes a lower bid than i. This probability is multiplied by the sum offered by bidder i.

Note that this expression of the expected utility is valid for at least four bidders. In order to study the three bidders case, it is necessary to (slightly) change the third line. To do this, we have to stop at the second line of the computation of the term B_I in the appendix. Thus, this term is written as $\alpha_i \sum_{l \neq i} \left(\int_{\mathbb{R}_+} \int_{x_i}^{x_l} x_k dF_k(x_k) dF_l(x_l) + x_i F_k(x_i)(1 - F_l(x_i)) \right)$, where k is neither i nor l.

Proposition 3. *In the second-price all-pay auction only two bidders, named i and j, among n participate actively. Bidder i's mixed strategy is given by the cumulative distribution* $F_i(x) = 1 - \exp\left(-\frac{1-2\alpha_j}{v_j}x\right) \forall x \in [0, +\infty)$ *and the expected revenue by* $\mathbb{E}R = \frac{2v_i v_j}{(1-2\alpha_i)v_j + (1-2\alpha_j)v_i}$.

Proof. See in Appendix ∎

The weakness of this result is that we do not know which bidders are going to participate. Thus, it might be that the two bidders with the highest values participate or the ones with the lowest values. This has some consequences on the expected revenue.

5 Revenue Comparisons

In this section, we investigate the performance of the revenues and the expected revenues obtained with the different auction designs.

We consider here that bidders have the same altruism level i.e. $\alpha_1 = \alpha_2 = \alpha$. Hence, the bidder with the highest value is also the one with the highest adjusted-value. The expected revenues become

$$\mathbb{E}R^{AP1} = \frac{1}{2}\frac{v_2}{1-\alpha}\left(\frac{v_2}{v_1} + 1\right) \text{ et } \mathbb{E}R^{AP2} = \frac{2}{1-2\alpha}\frac{v_i v_j}{v_i + v_j} \quad i, j \in N$$

Indices AP_i and WP_i correspond to i^{st}-price all-pay and winner-pay auctions. If bidders are thoroughly altruistic, i.e. $\alpha^{AP1} \longrightarrow 1$ and $\alpha^{AP2} \longrightarrow 1/2$, the expected revenues diverge as Goeree et al. (2005) predicted. Thus, the altruism level is an essential element to determine the expected revenue. When bidders' altruism levels are the same, the rent of the auction is at least equal to the rent one would obtain with non-altruistic bidders.

In the following, we compare our results on all-pay auctions with externalities to Ettinger's (2002) results on winner-pay auctions with externalities. These results are summed up in table 1:

$v_1 > v_2 > v_3 > v_i \ \forall i > 3$	R^{WP1}	R^{WP2}	$\mathbb{E}R^{AP1}$	$\mathbb{E}R^{AP2}$
$\alpha > 0$	v_2	v_1	$\frac{1}{2}\frac{v_2}{1-\alpha}\left(\frac{v_2}{v_1}+1\right)$	$\frac{2}{1-2\alpha}\frac{v_1 v_i}{v_1+v_i},\ i \neq 1$
$\alpha = 0$	v_2	v_2	$\frac{v_2}{2}\left(\frac{v_2}{v_1}+1\right)$	$2\frac{v_1 v_i}{v_1+v_i},\ i \neq 1$

Table 1: Revenues and expected revenues

From the *revenue equivalence principle*, we know that all the auction designs with homogeneous values and without any externalities lead to the same revenue. Moreover, if we consider homogeneous values with charity components (externalities), we find the same qualitative results than Goeree et al. (2005). Then, externalities improve the revenue performance of the all-pay auctions relatively to the winner-pay auctions. On the contrary, in a framework with no externality and heterogeneous bidders, winner-pay auctions outperform the first-price all-pay auction and could outperform the second-price all-pay auction (this depends which bidders are going to participate). Then, the asymmetry component improves the revenue performance of the winner-pay auctions relative to the first-price all-pay auctions and could improve it relative to the second-price all-pay auction. Hence, asymmetry and charity have opposite effects on the revenue comparaison among all-pay and winner-pay auctions. Thus, the revenue comparison result with charitable and asymmetric bidders is not obvious.

Moreover, if our framework is suited to charity dinners with complete information (for example dinners held by a local Rotary Club), first-price and second-price all-pay auctions contradict Goeree et al.'s (2005) qualitative results. In order to analyze the impact of asymmetry on rents, we use the following definition.

Definition. *The level of asymmetry between bidders' valuations will be considered very high if* $v_1 - v_2 > 2\alpha(v_1 + v_2)$*, high if* $v_1 - v_2 > 2\alpha v_1$*, low if* $v_1 - v_2 < 2\alpha v_1 - v_1 + v_2\frac{v_2}{v_1}$ *and medium if* $2\alpha v_1 > v_1 - v_2 > 2\alpha v_1 - v_1 + v_2\frac{v_2}{v_1}$*.*

Proposition 4. *We assume that* $\alpha_i = \alpha \ \forall i$ *and that the bidder with the highest value takes part in the second-price all-pay auction. Then, this design raises the highest revenue such that* $\mathbb{E}R^{AP2} > R^{WP2}$ *if and only if the level of asymmetry between valuations is not very high,* $\mathbb{E}R^{AP2} > R^{WP1}$ *and* $\mathbb{E}R^{AP2} > \mathbb{E}R^{AP1}$ *independently of the level of asymmetry.*

All other things being equal, $\mathbb{E}R^{AP1} > R^{WP2}$ *if and only if the level of asymmetry between valuations is low,* $R^{WP2} > \mathbb{E}R^{AP1} > R^{WP1}$ *if and only if this level is medium, and* $R^{WP1} > \mathbb{E}R^{AP1}$ *if and only if it is high.*

Proof. Computations. ∎

The second-price all-pay auction generates a higher rent than the second-price winner-pay auction if and only if the level of asymmetry is not very high and a higher rent than all the other auction designs as long as the bidder with the highest adjusted-value takes part in the auction. Moreover, the revenue performance of the second-price all-pay auctions can be interpreted in another way when the bidder with the highest adjusted-value participates in the auction. Given v_1 and v_2, the second-price all-pay auction outperfoms the second-price winner-pay auction when the bidders' altruism level is superior to $\frac{1}{2}\frac{v_1 - v_2}{v_1 + v_2}$.

On the contrary, when this bidder does not take part in the auction, the ranking of the expected revenue raised in the second-price all-pay auction depends on the asymmetry between bidders' valuations.

As for second-price all-pay auctions, we can interpret the revenue performance of the first-price all-pay auction relatively to the winner-pay auctions in two independent ways.

- First of all, given the altruism level α, the (first-price) all-pay auction is dominated by the first-price winner-pay auction when asymmetry is high. Furthermore, this all-pay auction raises more money than the second-price winner-pay auction when asymmetry is low. Thus, in order to determine which design is better to raise money for charity, we need to know the level of asymmetry between bidders.

- Given v_1 and v_2, the (first-price) all-pay auction is dominated by first and second-price winner-pay auctions when the bidders' altruism level is less than $\frac{1}{2}(1-\frac{v_2}{v_1})$. Yet, the all-pay auction outperforms the first-price auction and is dominated by the second-price auction when the bidders' altruism level is inferior to $1-\frac{1}{2}\frac{v_2}{v_1}(\frac{v_2}{v_1}+1)$ and superior to $\frac{1}{2}(1-\frac{v_2}{v_1})$. In particular, the threshold above which this all-pay auction raises more money than the first-price winner-pay auction is less than $\frac{1}{2}$. Lastly, the first-price all-pay auction outperforms the winner-pay auctions when $\alpha > 1-\frac{1}{2}\frac{v_2}{v_1}(\frac{v_2}{v_1}+1)$.

The greater the asymmetry, the higher the level of altruism needs to be for the first-price all-pay auction to give a higher rent than the winner-pay auctions and for the second-price all-pay auction to give a higher rent than the second-price winner-pay auction. The difference between the expected revenue of all-pay auctions and the revenue of winner-pay auctions are depicted in Figure 1, 2 and 3. These figures show the limits (in terms of rent domination) for the first-price and second-price all-pay auctions. We use two parameters: altruism level and the asymmetry among bidders' values (from left to right, $\dfrac{v_2}{v_1}$ varies from 0.9 to its limit in zero with a 0.1 step).

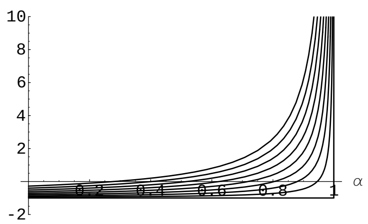

Figure 1: $\mathbb{E}R^{AP1} > R^{WP2}$

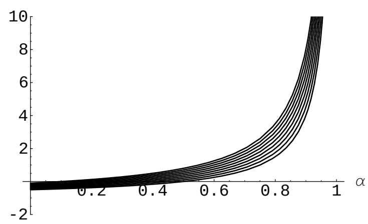

Figure 2: $\mathbb{E}R^{AP1} > R^{WP1}$

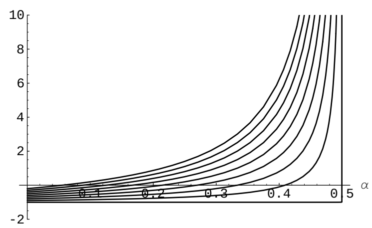

Figure 3: $\mathbb{E}R^{AP2} > R^{WP2}$

As a consequence, in order to determine which design is better to raise money for charity we need to know both levels of asymmetry and altruism. Contrary to the results of Goeree et al. (2005), here the all-pay auctions do not always outperform the winner-pay auctions.

6 Conclusion

This paper shows that all-pay auctions do not always raise higher revenue for charity than winner-pay auctions. This result depends on the asymmetry between bidders. In particular, winner-pay auctions outperform first-price all-pay auction when the asymmetry between bidders is strong. This contradicts Goeree et al. (2005)'s results. Our work can be related to the one of Carpenter et al. (2008). Indeed, they have found in a field experiment that the first-price winner-pay auction outperforms the first-price all-pay auction. This could be due to a strong asymmetry between bidders.

This work could be completed by a laboratory experiment. To date, only one lab experiment (Onderstal and Schram (2009)) has been led on charity auctions, with opposite results to the Carpenter et al. (2008) field experiment. Onderstal and Schram (2009) find similar results to Goeree et al. (2005). However, our results are quite different from Goeree et al. (2005)'s because of the introduction of asymmetric valuations and the information setting. That is why it would be interesting to test our prediction with the introduction of asymmetry between the bidders' valuation: all-pay auctions can be dominated by winner-pay auctions. Finally, theoretical and experimental works should be led to the form of the externalities which are mainly considered in the litterature as linear.

All-pay auctions with externalities that are independent of the winner's identity but are functions of the amount raised have other applications in economics.
One application is team theory. This illustration could be connected to other forms of team work (particularly in firms) leading to social promotion. Let us consider, a team sport like basketball. Every year during the American basketball championship (NBA) and the all-stars game finals, the most valuable player (MVP) is elected. During such games, every player makes the highest effort to win the event but also to be elected MVP of the game. Each player benefits from the team's effort to win the game and thus can be elected MVP thanks to the externality of the total efforts made. v_i represents the player's value for the MVP title. Therefore, her effort x_i has two goals: to win the game and be elected MVP. When a player is not elected MVP, she takes advantage of the externality by winning the game. As a player endeavors to win the game by making the highest effort, she also helps her team-mates to be elected MVP.

Appendix

Proof of Proposition 2. The sketch of this proof follows the same logic as the proof of Proposition 2 in Anderson et al. (1998). For the same reasons as those pointed out without externalities in Baye et al. (1996), the two players make their bids on the common support $[0, b]$ and the density function, $F_i' = f_i$ exists. The set of equilibria in mixed strategies is completely characterized by a Nash equilibria where only pure strategies which are better responses to the other strategies are played with a strictly positive probability. All these strategies lead to the same expected utility. From now on, we denote $\lambda_i = \frac{1}{v_i}$. Let us denote $\boldsymbol{F}(x)$ the vector of mixed strategies with $F_i(x)$ the i^{th} component. Let T be an operator such as $T : \boldsymbol{F}(x) \longmapsto \boldsymbol{TF}(x)$ with components

$$TF_i(x) \equiv \lambda_j x - \lambda_j \int_0^b h_j(x, y) f_i(y) dy + \text{ constant} \tag{4}$$

As F_i is a continuous function, we restrict our study to the set of continuous functions on $[0, b]$ denoted $C[0, b]$. In particular, we consider $D_i = \{F_i \in C[0, b] \backslash ||F_i|| \leq 1\}$ with $||.||$ the supremum norm. The set $\boldsymbol{D} \equiv D_1 \times D_2$, which includes all the continuous distribution functions, is closed and convex but not compact (as it is an infinite dimensional set). The set $\boldsymbol{D}$ is endowed with the norm $||\boldsymbol{F}||_2 = \max_{i=1,2} ||F_i||$. Thus, to prove that a vector $F_i(x)$ is solution of (4) for $i = 1, 2$ we apply the following Schauder's second theorem:

Theorem 1 (Schauder, 1930)**.** *If $\boldsymbol{D}$ is a closed convex subset of a normed space and $\boldsymbol{E}$ a* *relatively compact subset of* $\boldsymbol{D}$*, then every continuous mapping of* $\boldsymbol{D}$ *to* $\boldsymbol{E}$ *has a fixed-point.*

To apply this theorem, we need to prove two parts. First, that $\boldsymbol{E} \equiv \{\boldsymbol{TF} \backslash \boldsymbol{F} \in \boldsymbol{D}\}$ is relatively compact.[12] Second, T is a continuous mapping from $\boldsymbol{D}$ to $\boldsymbol{E}$.

In order to establish that $\boldsymbol{E}$ is relatively compact, the characterization of relative compactness in the space of continuous functions given by the Arzelà-Ascoli's theorem is used.

Theorem 2 (Arzelà-Ascoli, 1895)**.** *A set of functions in* $C[0,b]$*, with the supremum norm, is relatively compact if and only if it is uniformly bounded and equicontinuous on* $[0,b]$*.*

Thus, show that $\boldsymbol{E}$ is relatively compact is equivalent to showing that $\boldsymbol{E}$ is uniformly bounded and equicontinuous on $[0,b]$. Generalization of the assumption $A2$ leads to $\frac{\partial h_i}{\partial x}(x,y) < 1$ for all $y \in [0,b]$ and $i = 1,2$. Then, $TF_i(x)$ is nondecreasing. Besides, $|TF_i(x)| \leq TF_i(b) = 1$, for all $x \in [0,b]$, $F_i \in D_i$ and $i = 1,2$. Consequently, $\boldsymbol{E}$ is uniformly bounded. Let us show that $\boldsymbol{E}$ is equicontinuous. We need to show that$\forall \varepsilon, \exists \eta$ such that $|TF_i(x_1) - TF_i(x_2)| < \varepsilon$ when $|x_1 - x_2| < \eta \ \forall F_i \in D_i$ and $i = 1,2$.

$$
\begin{aligned}
|TF_i(x_1) - TF_i(x_2)| &= \left| \lambda_j(x_1 - x_2) - \lambda_j \int_0^b [h_j(x_1,y) - h_j(x_2,y)] f_i(y) dy \right| \\
&\leq \lambda_j \left[|x_1 - x_2| + \left| \int_0^b [h_j(x_1,y) - h_j(x_2,y)] f_i(y) dy \right| \right] \\
&\leq \lambda_j |x_1 - x_2| \left[1 + \frac{|\sup_{y\in[0,b]}[h_j(x_1,y) - h_j(x_2,y)]|}{|x_1 - x_2|} \right] \\
&< \lambda_j \eta \left[1 + \frac{|\sup_{y\in[0,b]}[h_j(x_1,y) - h_j(x_2,y)]|}{|x_1 - x_2|} \right].
\end{aligned}
$$

The function h_j is continuous and bounded on $[0,b]$. $[0,b]$ is a compact set which explains the result of the last line. Denoted $\kappa_j \equiv |\sup_{y\in[0,b]}[h_j(x_1,y) - h_j(x_2,y)]|$. Thus, $|TF_i(x_1) - TF_i(x_2)| < \varepsilon$ for $\eta = \varepsilon \min_{j=1,2} \dfrac{|x_1 - x_2|}{\lambda_j(|x_1 - x_2| + \kappa_j)}$ for all $F_i \in D_i$ and $i = 1,2$.

Now, let us prove the continuity of T. Operator T is continuous if, for all $\boldsymbol{F}^1, \boldsymbol{F}^2$ and for all $\varepsilon > 0$, there exists a $\eta > 0$ such that $||\boldsymbol{TF}^1(x) - \boldsymbol{TF}^2(x)||_2 < \varepsilon$ when $||\boldsymbol{F}^1 - \boldsymbol{F}^2||_2 < \eta$. Let us write $F_i^1(x) = F_i^2(x) + g_i(x)$ with $-\eta < g_i(x) < \eta \ \forall x \in [0,b]$ and $i = 1,2$. Henceforth

$$
\begin{aligned}
|TF_i^1(x) - TF_i^2(x)| &= \left| -\lambda_j \int_0^b h_j(x,y)(f_i^1(y) - f_i^2(y)) dy \right| \\
&\leq \lambda_j \int_0^b |h_j(x,y)||g_i'(y)| dy \\
&\leq h_j(b,b) \lambda_j \int_0^b |g_i'(y)| dy \\
&< h_j(b,b) \lambda_j \eta.
\end{aligned}
$$

To go from the first to the second line, notice that $f_i^1(x) - f_i^2(x) = g_i'(x)$. We use the fact that h_j is a continuous function on $[0,b]$ bounded by a maximum $h_j(b,b)$ to go to the third line. Hence, $||\boldsymbol{TF}^1(x) - \boldsymbol{TF}^2(x)||_2$ is inferior to $\varepsilon > 0$ when $\eta = \min_{j=1,2} \dfrac{\varepsilon}{\lambda_j h_j(b,b)}$ for all $x \in [0,b]$. ∎

[12] A space is relatively compact when its closed span is compact.

Proof of Proposition 3.

Lemma 1. *Let us consider i and j the two potential participants. Then, bidder i's mixed strategy is given by $F_i(x) = 1 - \exp\left(-\frac{1-2\alpha_j}{v_j}x\right)$ $\forall x \in [0, +\infty)$ and the expected revenue by $\mathbb{E}R = \frac{2v_i v_j}{(1-2\alpha_i)v_j + (1-2\alpha_j)v_i}$.*

Proof. The expected utility of bidder i is given by

$$\mathbb{E}U_i(x_i, X_j) = \int_0^{x_i} (v_i - (1-2\alpha_i)x)dF_j(x) - (1-2\alpha_i)x_i(1 - F_j(x_i))$$

Then, dividing the expected utility by $1-2\alpha_i$ and considering the adjusted values $\frac{v_i}{1-2\alpha_i}$ instead of the values v_i we get a positive transformation of the expected utility without any externalities. The mixed strategies at the equilibrium would be not alterated by this transformation.

Lemma 2. *Let n be the number of potential participants. Then, only two bidders among n participate actively to the auction.*

Proof. By (2) we have the expected utility:

$$\mathbb{E}U_i(x_i, \boldsymbol{X}_{-i}) = v_i \prod_{j\neq i} dF_j(x_j) - (1-\alpha_i) \underbrace{\int_{\mathbb{R}_+^{n-1}} t_i(\boldsymbol{x}) \prod_{j\neq i} dF_j(x_j)}_{A} + \alpha_i \underbrace{\int_{\mathbb{R}_+^{n-1}} \sum_{j\neq i} t_j(\boldsymbol{x}) \prod_{j\neq i} dF_j(x_j)}_{B}$$

A represents bidder i's expected payment when we take into account her own external effect. The term B is the expected payment of bidder i's rivals. $\alpha_i B$ is the sum of the externalities of bidder i's rivals from which i benefits.

We can write A again as follows

$$\underbrace{\int_{\mathbb{R}_+^{n-1}} x^{(2)} \mathbb{1}_{\substack{x_i \geq x_j \\ \forall j\neq i}} \prod_{j\neq i} dF_j(x_j)}_{A_I} + \underbrace{\int_{\mathbb{R}_+^{n-1}} x_i \mathbb{1}_{\substack{\exists k / x_k > x_i \\ k\neq i}} \prod_{j\neq i} dF_j(x_j)}_{A_{II}}.$$

The term A_I is i's expected payment when she wins *i.e.* he pays the second highest bid. A_{II} is i's expected payment when she loses. She could then either be the second highest bidder or a lower bidder.

$$\begin{aligned}
A_I &= \int_{\mathbb{R}_+^{n-1}} \sum_{j\neq i} x_j \mathbb{1}_{\substack{x_k \leq x_j \leq x_i \\ \forall k \neq \{j,i\}, j\neq i}} \prod_{j\neq i} dF_j(x_j) \\
&= \int_{\mathbb{R}_+} \sum_{j\neq i} x_j \mathbb{1}_{x_j \leq x_i} \left\{ \int_{\mathbb{R}_+^{n-2}} \prod_{k\neq i,j} \mathbb{1}_{x_k \leq x_j \leq x_i} \prod_{k\neq i,j} dF_k(x_k) \right\} dF_j(x_j) \\
&= \int_{\mathbb{R}_+} \sum_{j\neq i} x_j \mathbb{1}_{x_j \leq x_i} \left\{ \prod_{k\neq i,j} \int_{\mathbb{R}} \mathbb{1}_{x_k \leq x_j \leq x_i} dF_k(x_k) \right\} dF_j(x_j) \\
&= \int_{\mathbb{R}_+} \sum_{j\neq i} x_j \mathbb{1}_{x_j \leq x_i} \prod_{k\neq i,j} F_k(x_j) dF_j(x_j) \\
&= \int_0^{x_i} x dG_i(x)
\end{aligned}$$

We get the first line from the fact that $x^{(2)}\mathbb{1}_{x_i \geq x_j} = \sum_{j \neq i} x_j \mathbb{1}_{\substack{x_k \leq x_j \leq x_i \\ \forall k \neq \{j,i\}, j \neq i}}$. The independence of the distribution functions explains how we go from the second to the third line. By denoting $dG_i(x) = \sum_{j \neq i} \prod_{k \neq i,j} F_k(x) dF_j(x)$, we obtain the final result.

$$
\begin{aligned}
A_{II} &= \int_{\mathbb{R}_+^{n-1}} x_i (1 - \mathbb{1}_{i \in Q(x)}) \prod_{j \neq i} dF_j(x_j) \\
&= x_i - x_i \prod_{j \neq i} F_j(x_i) \\
&= x_i (1 - G_i(x_i))
\end{aligned}
$$

The independence of the distribution functions, explains how we go from the first line to the second.

B can be written also like

$$
\begin{aligned}
B &= \sum_{l \neq i} \int_{\mathbb{R}_+^{n-1}} t_l(\boldsymbol{x}) \prod_{j \neq i} dF_j(x_j) \\
&= \sum_{l \neq i} \Bigg\{ \underbrace{\int_{\mathbb{R}_+^{n-1}} x^{(2)} \mathbb{1}_{\substack{x_l \geq x_k \\ \forall k \neq l}} \prod_{j \neq i} dF_j(x_j)}_{B_I} + \underbrace{\int_{\mathbb{R}_+^{n-1}} x_l \mathbb{1}_{\substack{\exists k / x_l < x_k \\ k \neq l}} \prod_{j \neq i} dF_j(x_j)}_{B_{II}} \Bigg\}
\end{aligned}
$$

We add all the expected external effects. The case where player $l \neq i$ takes the second highest bid is distinguished from the others.

$$
\begin{aligned}
B_I &= \int_{\mathbb{R}_+^{n-1}} \sum_{k \neq l} x_k \mathbb{1}_{\substack{x_m \leq x_k \leq x_l \\ \forall m \neq \{k,l\}}} \prod_{j \neq i} dF_j(x_j) \\
&= \int_{\mathbb{R}_+^{n-1}} \sum_{k \neq l} x_k \prod_{m \neq \{k,l\}, k \neq l} \mathbb{1}_{x_m \leq x_k \leq x_l} \prod_{j \neq i} dF_j(x_j) \\
&= \int_{\mathbb{R}_+^{n-1}} \sum_{k \neq i,l} x_k \prod_{\substack{m \neq i,k,l \\ k \neq l}} \mathbb{1}_{x_m \leq x_k \leq x_l} dF_m(x_m) \mathbb{1}_{x_i \leq x_k \leq x_l} dF_k(x_k) dF_l(x_l) \\
&\quad + \int_{\mathbb{R}_+^{n-1}} x_i \prod_{m \neq i,l} \mathbb{1}_{x_m \leq x_i \leq x_l} \prod_{j \neq i} dF_j(x_j) \\
&= \int_{\mathbb{R}_+^2} \sum_{k \neq i,l} x_k \int_{\mathbb{R}_+^{n-3}} \prod_{\substack{m \neq i,k,l \\ k \neq l}} \mathbb{1}_{x_m \leq x_k} dF_m(x_m) \mathbb{1}_{x_i \leq x_k \leq x_l} dF_k(x_k) dF_l(x_l) \\
&\quad + x_i \int_{\mathbb{R}_+} \prod_{m \neq i,l} \left\{ \int_0^{x_i} dF_m(x_m) \right\} \mathbb{1}_{x_i \leq x_l} dF_l(x_l) \\
&= \int_{\mathbb{R}_+^2} \sum_{k \neq i,l} x_k \prod_{\substack{m \neq i,k,l \\ k \neq l}} F_m(x_k) \mathbb{1}_{x_i \leq x_k \leq x_l} dF_k(x_k) dF_l(x_l) + x_i \prod_{m \neq i,l} F_m(x_i)(1 - F_l(x_i)) \\
&= \int_{\mathbb{R}_+} \int_{x_i}^{x_l} \sum_{k \neq i,l} x_k \prod_{\substack{m \neq i,k,l \\ k \neq l}} F_m(x_k) dF_k(x_k) dF_l(x_l) + x_i \left(\prod_{m \neq i,l} F_m(x_i) - G_i(x_i) \right).
\end{aligned}
$$

$$
\begin{aligned}
B_{II} &= \int_{\mathbb{R}_+^{n-1}} x_l(1-\mathbb{1}_{l\in Q(x)}) \prod_{j\neq i} dF_j(x_j) \\
&= \int_{\mathbb{R}_+^{n-1}} x_l \prod_{j\neq i} dF_j(x_j) - \int_{\mathbb{R}_+^{n-1}} x_l \prod_{k\neq i,l} \Big(\mathbb{1}_{x_k\leq x_l} dF_k(x_k)\Big) \mathbb{1}_{x_i\leq x_l} dF_l(x_l) \\
&= \int_{\mathbb{R}_+^{n-1}} x_l \prod_{j\neq i} dF_j(x_j) - \int_{\mathbb{R}_+} x_l \mathbb{1}_{x_i\leq x_l} \Big\{ \int_{\mathbb{R}_+^{n-2}} \prod_{k\neq i,l} \mathbb{1}_{x_k\leq x_l} dF_k(x_k) \Big\} dF_l(x_l) \\
&= \int_{\mathbb{R}_+} x_l dF_l(x_l) - \int_{\mathbb{R}_+} x_l \mathbb{1}_{x_i\leq x_l} \prod_{k\neq i,l} F_k(x_l) dF_l(x_l) \\
&= \int_{\mathbb{R}_+} x_l (1-\mathbb{1}_{x_i\leq x_l} \prod_{k\neq i,l} F_k(x_l)) dF_l(x_l).
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\mathbb{E}U_i(x_i, \boldsymbol{X}_{-i}) = &\int_0^{x_i} (v_i - (1-\alpha_i)x) dG_i(x) - (1-\alpha_i)x_i(1-G_i(x_i)) \\
&+ \alpha_i \sum_{l\neq i} \int_{\mathbb{R}_+} x_l (1-\mathbb{1}_{x_i\leq x_l} \prod_{k\neq i,l} F_k(x_l)) dF_l(x_l) \\
&+ \alpha_i \sum_{l\neq i} \Bigg(\int_{\mathbb{R}_+} \int_{x_i}^{x_l} \sum_{k\neq i,l} x_k \prod_{\substack{m\neq i,k,l \\ k\neq l}} F_m(x_k) dF_k(x_k) dF_l(x_l) + x_i \prod_{m\neq i,l} F_m(x_i)(1-F_l(x_i)) \Bigg).
\end{aligned}
$$

Next, we note:

$$
G_{il}(x) = \prod_{k\neq i,l} F_k(x) \text{ et } G'_{il}(x) = \sum_{j\neq i,l} \prod_{k\neq i,l,j} F_k(x) dF_j(x).
$$

As the expected utility is constant at the equilibrium, the FOC leads to

$$
v_i G'_i(x) - (1-\alpha_i)(1-G_i(x)) + \alpha_i \sum_{l\neq i} G_{il}(x) - \alpha_i \sum_{l\neq i} G_{il}(x) F_l(x) - \alpha_i x \sum_{l\neq i} G'_{il}(x) F_l(x) = 0.
$$

Notice that $(n-1)G_i(x) = \sum_{l\neq i} G_{il}(x)F_l(x)$ and $(n-2)G'_i(x) = \sum_{l\neq i} G'_{il}(x)F_l(x)$ henceforth:

$$
(v_i - \alpha_i x(n-2))G'_i(x) + (1-\alpha_i n)G_i(x) = (1-\alpha_i) - \alpha_i \sum_{l\neq i} G_{il}(x) \quad \forall i \in \{1, ..., n\}. \tag{A1}
$$

This result is true for all $n > 3$. The closed characterization of the solution is very difficult. Yet, we can deduce the solution by an alternative way. Indeed, let F_i and F_j be the mixed strategies of the two bidders i and j. We can notice that the derivative of the expected utility of a third bidder k $H_k(x) = \frac{\partial \mathbb{E}U_k}{\partial x}(x_i, X_1, X_2)$ is a monotonous increasing function. Furthermore, $H_k(0) = -(1-\alpha_k)$ and $\lim_{x\to+\infty} H_k(x) = 0$. Thus, given the mixed strategies of i and j, k do not participate. This result can easily be extended to a number n of bidders. For that, we should use recurrence.

From Lemma 1 and 2 the result follows. ■

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